

ON THE REPRESENTATION TYPES OF FINITE EI CATEGORIES

LIPING LI

ABSTRACT. A finite EI category is a small category with finitely many morphisms such that every endomorphism is an isomorphism. They include finite groups, finite posets and free categories of finite quivers as special cases. In this paper we consider the representation types of finite EI categories, describe some criteria for finite representation type, and use them to classify the representation types of several classes of finite EI categories with extra properties.

1. INTRODUCTION

The concept of *finite EI categories* is introduced and studied by Dieck, Lück, Webb, Xu, etc in [9, 17, 21, 22, 23, 24, 25]. They are small categories with finitely many morphisms such that every endomorphism is an isomorphism. Particular examples of finite EI categories include finite groups (viewed as finite EI categories with one object), finite posets, and free categories of finite acyclic quivers. Consequently, the representation theory of finite EI categories are more general than the representation theories of these structures.

Let k be an algebraically closed field and \mathcal{C} be a finite EI category. A *representation* of \mathcal{C} is a functor from \mathcal{C} to $k\text{-vec}$, the category of finite-dimensional k -vector spaces. A *morphism* between two representations of \mathcal{C} is precisely a natural transformation. The finite EI category \mathcal{C} determines a finite-dimensional k -algebra $k\mathcal{C}$ called the *category algebra*. By Theorem 7.1 of [19], the category of representations of \mathcal{C} is Morita equivalent to $k\mathcal{C}\text{-mod}$, the category of finitely generated $k\mathcal{C}$ -modules. The category \mathcal{C} is said to be of *finite representation type* if $k\mathcal{C}\text{-mod}$ has only finitely many indecomposable objects up to isomorphism. Otherwise, it is said to be of *infinite representation type*.

Although the classifications of representation types of finite groups, finite acyclic quivers (Gabriel's theorem) and finite posets (see [16]) have already been done a long time ago, the classification of representation types of finite EI categories is far from completion. Indeed, even the answer for finite EI categories with only two objects is still unknown. Some results on this topic can be found in [10, 11].

Intuitively, a (skeletal) finite EI category \mathcal{C} can be regarded as a combination of a finite poset and several finite groups (the automorphism groups of objects). Therefore, the representation types of the underlying poset and these finite groups are certainly among the factors determining the representation type of \mathcal{C} . Indeed,

2000 *Mathematics Subject Classification*. 16G10, 16G20.

Key words and phrases. finite EI categories, category algebras, representation types.

The author would like to thank his thesis advisor, Professor Peter Webb, for the proposal to study this problem, and for many invaluable suggestions in numerous discussions. He also express appreciation to Professor Edward Green, Professor Markus Schmidmeier and Dr. Ryan Kinser, from whom the author learned many useful techniques applied in this paper.

with some easy reduction we show that if \mathcal{C} is of finite representation type, then the underlying poset and all these finite groups should have finite representation type as well. However, the converse statement is obviously not true. Actually, non-endomorphisms in \mathcal{C} form a disjoint union of bisets (see [6, 21] for a description of bisets) under the actions of those automorphism groups, and the structure of these bisets also plays an essential role in determining the representation type of \mathcal{C} . In this paper we mainly focus on analyzing the structure of these bisets, and derive a lot of useful criteria on the representation type of \mathcal{C} .

In the case that the automorphism groups of all objects have orders invertible in k , we can construct the ordinary quiver \tilde{Q} (which is an acyclic finite quiver) of $k\mathcal{C}$ using the algorithm described in [14]. Therefore, the category algebra $k\mathcal{C}$ is isomorphic to a quotient algebra of the hereditary algebra $k\tilde{Q}$. In particular, when \mathcal{C} is a *free EI category* (defined in the next section), $k\mathcal{C} \cong k\tilde{Q}$ and its representation type can be determined by Gabriel's theorem.

If \mathcal{C} has an object whose automorphism group has non-invertible order in k , the situation becomes much more complicated. For finite EI categories with two objects whose automorphism groups are p -groups, where $p = \text{char}(k)$, the classification has been obtained in [11] by applying Bongartz's list described in [3, 5]. We consider the case which is a little bit more general, i.e., arbitrary finite EI categories for which the automorphism groups of all objects are p -groups. In particular, for $p \geq 5$, we classify the representation types of this class of finite EI categories with three objects.

In another special case we consider finite EI categories with two objects and arbitrary automorphism groups. Since a finite EI category is of finite representation type if and only if so are all full subcategories, the representation types of full subcategories with two objects can provide us a lot useful information.

We introduce some notation here. Let \mathcal{C} be a finite EI category pictured as below, where $G = \mathcal{C}(x, x)$ and $H = \mathcal{C}(y, y)$. It is easy to see that if \mathcal{C} is of finite representation type, then $\mathcal{C}(x, y)$ has only one orbit as an (H, G) -biset. Thus we let α be a fixed morphism in $\mathcal{C}(x, y)$. Define $G_0 = \text{Stab}_G(\alpha)$, $G_1 = \text{Stab}_G(H\alpha)$, $H_0 = \text{Stab}_H(\alpha)$, and $H_1 = \text{Stab}_H(\alpha G)$. Then $G_0 \triangleleft G_1 \leq G$ and $H_0 \triangleleft H_1 \leq H$. Moreover, $G_1/G_0 \cong H_1/H_0$ as groups.

$$\mathcal{C} : \quad \begin{array}{c} \textcircled{G} \\ \curvearrowright \end{array} x \xrightarrow[\dots]{H\alpha G} y \begin{array}{c} \curvearrowleft \\ \textcircled{H} \end{array}.$$

Note that \mathcal{C} is clearly a finite free EI category. We characterize its representations explicitly by linear maps satisfying some property related to representations of the above groups. Therefore, techniques as biset decompositions, inductions and restrictions, and group representations (for example, permutation modules, block theory and Brauer graphs, etc.) can be applied to exploit its representation type. Particularly we get the following criterion:

Theorem 1.1. *Let \mathcal{C} be as above and $p = \text{char}(k) \geq 0$. If \mathcal{C} is of finite representation type, then the following conditions must be satisfied:*

- (1) *Both G and H have cyclic Sylow p -subgroups.*
- (2) *Either G or H acts transitively on $\mathcal{C}(x, y)$, i.e., $\mathcal{C}(x, y)$ equals $H\alpha$ or αG .*
- (3) *If $p \geq 5$, then either $Op'G$ or $Op'H$ acts trivially on $\mathcal{C}(x, y)$. Moreover, for every p -subgroup $P \leq G$ (resp., $Q \leq H$), $P \cap G_0$ is either 1 or P (resp., $Q \cap H_0$ is either 1 or Q).*

- (4) If $p \geq 17$, then normal Sylow p -subgroups (if exist) of G and H act trivially on $\mathcal{C}(x, y)$.

Consequently, in many cases the representation type of \mathcal{C} can be determined from two pieces of information: the transitivity of actions by G and H , and the triviality of actions by Sylow p -subgroups in G and H . Indeed, the combination of conditions (a-c)

- (a) Both G and H act transitively.
- (b) One of G and H acts transitively, and the other one does not.
- (c) Neither G nor H acts transitively.

and conditions (1-3)

- (1) Both $O^{p'}G$ and $O^{p'}H$ act trivially.
- (2) One of $O^{p'}G$ and $O^{p'}H$ acts trivially, and the other one does not.
- (3) Neither $O^{p'}G$ nor $O^{p'}H$ acts trivially.

gives us 8 situations (the combination (a)+(2) cannot happen). By Theorem 1.1, if $\text{char}(k) = p \neq 2, 3$, we get infinite representation type for five cases: (c)+(1), (c)+(2), (c)+(3), (a)+(3), (b)+(3). The following theorem tells us the finite representation type for case (a)+(1).

Theorem 1.2. *Let \mathcal{C} be as in the previous theorem, and suppose that both G and H act transitively on $\mathcal{C}(x, y)$. If $\text{char}(k) = p \neq 2, 3$, then \mathcal{C} is of finite representation type if and only if both $O^{p'}G$ and $O^{p'}H$ act trivially on $\mathcal{C}(x, y)$.*

Although cases (b)+(1) and (b)+(2) are not completely resolved here, by considering the structure of certain permutation modules we are able to obtain the following criteria, which tell us a lot information on the structure of the double cosets $H_1 \backslash H / H_1$.

Theorem 1.3. *Let \mathcal{C} be as in Theorem 1.1 and suppose that H acts transitively on $\mathcal{C}(x, y)$. If \mathcal{C} is of finite representation type, then for every simple summand S of $\text{Top}(k \uparrow_{H_0}^{H_1})$, the following conditions must be satisfied:*

- (1) $\text{Top}(S \uparrow_{H_1}^H)$ has at most three simple summands, and all these summands are non-isomorphic;
- (2) Every indecomposable summand M of $S \uparrow_{H_1}^H$ is uniserial or biserial, and if M is biserial, it is projective;
- (3) If $M \not\cong N$ are indecomposable summands of $S \uparrow_{H_1}^H$, then $\text{Hom}_{kH}(M, N) = 0$;
- (4) $\dim_k \text{End}_{kH}(k \uparrow_{H_1}^H) = |H_1 \backslash H / H_1| \leq 3$.

In particular, if both G and H are abelian, we get the following classification.

Theorem 1.4. *Let \mathcal{C} be as in Theorem 1.1 and $\text{char}(k) = p \neq 2, 3$. Without loss of generality assume that H acts transitively on $\mathcal{C}(x, y)$. Let $s = |O^{p'}G|$, $t = |O^{p'}H|$, and $n = |H : H_1|$. If both G and H are abelian, then \mathcal{C} is of finite representation type if and only if both Sylow p -subgroups $O^{p'}G$ and $O^{p'}H$ act trivially on $\mathcal{C}(x, y)$, and one of the following conditions holds:*

- (1) $n = 1$ for $s, t \geq p$, i.e., $H = H_1$;
- (2) $n \leq 2$ for $s = 1, t \geq p$ or $t = 1, s \geq p$;
- (3) $n \leq 3$ for $t = s = 1$.

We then turn to finite free EI categories. Since every finite EI category \mathcal{C} is quotient category of its *free EI cover* $\hat{\mathcal{C}}$, the finite representation type of $\hat{\mathcal{C}}$ implies the finite representation type of \mathcal{C} . Let \mathcal{C} be a connected, skeletal finite free EI category. We show that its *underlying quiver* (defined in Section 2) Q is a Dynkin quiver given that \mathcal{C} is of finite representation type. We consider the behavior of objects $x \in \text{Ob}\mathcal{C}$ with two or three *adjacent objects* (see Section 8), i.e., there are two or three *representative unfactorizable morphisms* (see Section 2) starting or ending at x . Finally, we obtain the following criterion:

Theorem 1.5. *Let \mathcal{C} be a connected, skeletal finite free EI category with finite representation type. Then the following conditions must hold:*

- (1) *The underlying quiver of \mathcal{C} is a Dynkin quiver.*
- (2) *For every object x in \mathcal{C} , $\mathcal{C}(x, x)$ has cyclic Sylow p -subgroups.*
- (3) *For each pair of distinct objects $x, y \in \text{Ob}\mathcal{C}$ such that $\mathcal{C}(x, y) \neq \emptyset$, either $\mathcal{C}(x, x)$ or $\mathcal{C}(y, y)$ acts transitively on $\mathcal{C}(x, y)$.*

Moreover, if $\text{char}(k) = p \neq 2, 3$, then:

- (4) *For each pair of distinct objects $x, y \in \text{Ob}\mathcal{C}$ such that $\mathcal{C}(x, y) \neq \emptyset$, either $O^{p'}\mathcal{C}(x, x)$ or $O^{p'}\mathcal{C}(y, y)$ acts trivially on $\mathcal{C}(x, y)$.*
- (5) *If there are two representative unfactorizable morphisms starting or ending at $x \in \text{Ob}\mathcal{C}$, then $\mathcal{C}(x, x)$ acts transitively on at least one of the bisets generated by them.*
- (6) *Let y be the unique object (if exists) at which three representative unfactorizable morphisms start or end. Then $\mathcal{C}(y, y)$ acts transitively on all bisets generated by them.*

Conditions (2-4) in this theorem actually holds for all finite EI categories.

This paper is organized as follows. Some background knowledge on finite EI categories and their representations is included in the second section. From the third section we begin to discuss their representation types. Some easy but useful reductions are described in Section 3. We consider in Section 4 finite EI categories for which the automorphism of all groups have invertible orders. Since results in this section are taken from [14], we omit most proofs.

In Sections 5-7 we study two special cases. The first special case, i.e., the automorphism groups of all objects are p -groups, is investigated in Section 5. The main result of this section is to classify the representation types of these categories with three objects for $p \geq 5$. In Section 6 we consider the second special case, i.e., finite EI categories with two objects. After characterizing explicitly their representations and developing several techniques such as biset decompositions and the corresponding inductions and restrictions, we study the actions by automorphism groups and their Sylow p -subgroups and prove Theorem 1.1. The classification of representation types of finite EI categories with two objects is developed in Section 7, where we determine the representation types for several special cases, and prove Theorems 1.2 and 1.3. In the last section we turn to general finite free EI categories, and prove the last theorem.

Throughout this paper k is an algebraically closed field and $k\text{-vec}$ is the category of finite-dimensional k -vector spaces. For a finite EI category \mathcal{C} , by $\text{Ob}\mathcal{C}$ and $\text{Mor}\mathcal{C}$ we denote sets of objects and morphisms in \mathcal{C} respectively; $\mathcal{C}(x, y)$ denotes the set of morphisms from x to y ; and by $\mathcal{C}\text{-rep}$ (or $k\mathcal{C}\text{-mod}$) we denote the category of representations of \mathcal{C} . For a module M , $\text{Top}(M)$ and $\text{Soc}(M)$ mean the top and

the socle of M respectively. All modules we consider are finitely generated left modules. Composition of morphisms and maps is from right to left. To deal with all characteristics in a uniform way, when $p = 0$, we view the trivial group 1 as the Sylow 0-subgroup of a finite group.

2. PRELIMINARIES

For the reader's convenience, we include in this section some background on the representation theory of finite EI categories. Please refer to [22, 23, 24, 25] for more details.

A *finite EI category* \mathcal{C} is a small category with finitely many morphisms such that every endomorphism in \mathcal{C} is an isomorphism. Examples of finite EI categories include finite groups (viewed as categories with one object), finite posets (all endomorphisms are identities), and *orbit categories* (see [23]). The category \mathcal{C} is *connected* if for any two distinct objects x and y , there is a list of objects $x = x_0, x_1, \dots, x_n = y$ such that either $\mathcal{C}(x_i, x_{i+1})$ or $\mathcal{C}(x_{i+1}, x_i)$ is not empty, $0 \leq i \leq n - 1$. Every finite EI category is a disjoint union of connected components, and each component as a full subcategory is again a finite EI category.

A *representation* of \mathcal{C} is a functor $R : \mathcal{C} \rightarrow k\text{-vec}$. The functor R assigns a vector space $R(x)$ to each object x in \mathcal{C} , and a linear transformation $R(\alpha) : R(x) \rightarrow R(y)$ to each morphism $\alpha : x \rightarrow y$ such that all composition relations of morphisms in \mathcal{C} are preserved under R . A *homomorphism* $\varphi : R_1 \rightarrow R_2$ of two representations is a natural transformation of functors.

A finite EI category \mathcal{C} determines a finite-dimensional k -algebra $k\mathcal{C}$ called the *category algebra*. It has a basis constituted of all morphisms in \mathcal{C} , and the multiplication is defined by the composition of morphisms (the composite is 0 when two morphisms cannot be composed) and bilinearity. By Theorem 7.1 of [19], a representation of \mathcal{C} is equivalent to a $k\mathcal{C}$ -module. Thus we do not distinguish these two concepts throughout this paper.

By Proposition 2.2 in [22], if \mathcal{C} and \mathcal{D} are equivalent finite EI categories, $k\mathcal{C}\text{-mod}$ is Morita equivalent to $k\mathcal{D}\text{-mod}$. Moreover, if $\mathcal{C} = \bigsqcup_{i=1}^m \mathcal{C}_i$ is a disjoint union of several full subcategories, then $k\mathcal{C} = k\mathcal{C}_1 \oplus \dots \oplus k\mathcal{C}_m$ as algebras. Thus it is sufficient to study the representation types of connected, skeletal finite EI categories. We make the following convention:

Convention: All finite EI categories in this paper are **connected** and **skeletal**. Thus endomorphisms, isomorphisms and automorphisms in a finite EI category coincide.

Under this hypothesis, if x and y are two distinct objects in \mathcal{C} with $\mathcal{C}(x, y) \neq \emptyset$, then $\mathcal{C}(y, x)$ is empty. Indeed, if this is not true, we can take $\alpha \in \mathcal{C}(y, x)$ and $\beta \in \mathcal{C}(x, y)$. The composite $\beta\alpha$ is an endomorphism of y , hence an automorphism. Similarly, the composite $\alpha\beta$ is an automorphism of x . Thus both α and β are isomorphisms, so x is isomorphic to y . But this is impossible since \mathcal{C} is skeletal and $x \neq y$.

We give here some illustrative examples showing that the representation theory of finite EI categories has applications in relevant fields such as representation theory of finite groups.

Example 2.1. Let \mathcal{C} be the following finite EI category such that: $\mathcal{C}(x, x) \cong \mathcal{C}(y, y) \cong G$ which is a finite group; $\mathcal{C}(x, y) \cong G$ on which $\mathcal{C}(x, x)$ acts by multiplication from the right side and $\mathcal{C}(y, y)$ acts by multiplication from the left side.

$$\mathcal{C} : \begin{array}{ccc} \textcircled{G} x & \xrightarrow{G G_G} & y \textcircled{G} \\ & \xrightarrow{\dots} & \end{array}$$

Then a representation \mathcal{C} is nothing but a kG -module homomorphism $\varphi : M \rightarrow N$.

Define another finite EI category \mathcal{D} as below, where: $\mathcal{D}(x, x) \cong \mathcal{D}(y, y) \cong \langle g \rangle$ is a cyclic group of order $p = \text{char}(k)$; $\mathcal{D}(x, y) = \{\alpha\}$ on which both automorphism groups act trivially.

$$\mathcal{D} : \begin{array}{ccc} \textcircled{\langle g \rangle} x & \xrightarrow{\alpha} & y \textcircled{\langle g \rangle} \end{array}$$

A representation of \mathcal{D} is a $k\langle g \rangle$ -module homomorphism $\varphi : M \rightarrow N$ such that $\varphi(\text{Top}(M)) \subseteq \text{Soc}(N)$.

Now we describe the concepts of *unfactorizable morphisms* and *finite free EI categories* (see [14]). These concepts will play an important role for our goal.

Definition 2.2. A morphism $\alpha : x \rightarrow z$ in a finite EI category \mathcal{C} is *unfactorizable* if α is not an isomorphism and whenever it has a factorization as a composite $x \xrightarrow{\beta} y \xrightarrow{\gamma} z$, then either β or γ is an isomorphism.

For finite EI categories, *unfactorizable morphisms* are precisely *irreducible morphisms* which are widely used in [2, 3, 24]. But in a more general context, they are different, see Example 2.4 in [14].

We describe some elementary properties of unfactorizable morphisms here.

Proposition 2.3. Let $\alpha : x \rightarrow y$ be an unfactorizable morphism in \mathcal{C} . Then $h\alpha g$ is also unfactorizable for $h \in \mathcal{C}(y, y)$ and $g \in \mathcal{C}(x, x)$. Moreover, any non-isomorphism in \mathcal{C} can be expressed as a composite of finitely many unfactorizable morphisms.

Proof. See Propositions 2.5 and 2.6 in [14]. \square

For an arbitrary finite EI category \mathcal{C} , the ways to decompose a non-isomorphism into unfactorizable morphisms need not be unique. For a type of special finite EI categories, this decomposition is unique up to trivial relations. That is, they satisfy the property defined below:

Definition 2.4. A finite EI category \mathcal{C} is called a *finite free EI category* if it satisfies the following *Unique Factorization Property (UFP)*: whenever a non-isomorphism α has two decompositions into unfactorizable morphisms,

$$x = x_0 \xrightarrow{\alpha_1} x_1 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_m} x_m = y$$

$$x = x_0 \xrightarrow{\beta_1} y_1 \xrightarrow{\beta_2} \dots \xrightarrow{\beta_n} y_n = y$$

then $m = n$, $x_i = y_i$, and there are $h_i \in \mathcal{C}(x_i, x_i)$ such that the following diagram commutes, $1 \leq i \leq n - 1$:

$$\begin{array}{ccccccc} x_0 & \xrightarrow{\alpha_1} & x_1 & \xrightarrow{\alpha_2} & \dots & \xrightarrow{\alpha_{n-1}} & x_{n-1} & \xrightarrow{\alpha_n} & x_n \\ \parallel \text{id} & & \downarrow h_1 & & \downarrow h_{\dots} & & \downarrow h_{n-1} & & \parallel \text{id} \\ x_0 & \xrightarrow{\beta_1} & x_1 & \xrightarrow{\beta_2} & \dots & \xrightarrow{\beta_{n-1}} & x_{n-1} & \xrightarrow{\beta_n} & x_n \end{array}$$

This definition of finite free EI categories is different from that in [14], where finite free EI categories are defined by finite EI quivers. However, by Proposition 2.8 in [14], these two definitions are equivalent.

Some properties of finite free EI categories are collected below.

Proposition 2.5. *Let \mathcal{C} be a finite EI category.*

- (1) *There is a finite free EI category $\hat{\mathcal{C}}$ and a full functor $\hat{F} : \hat{\mathcal{C}} \rightarrow \mathcal{C}$ such that \hat{F} is the identity map restricted to objects, automorphisms and unfactorizable morphisms in \mathcal{C} . This finite free EI category $\hat{\mathcal{C}}$ is unique up to isomorphism.*
- (2) *\mathcal{C} is a finite free EI category if and only if so are all full subcategories.*
- (3) *The algebra $k\mathcal{C}$ is hereditary if and only if \mathcal{C} is a finite free EI category and the automorphism group of each object has order invertible in k .*

Proof. See Propositions 2.9, 2.10 and Theorem 5.3 in [14]. The category $\hat{\mathcal{C}}$ in (1) is called the *free EI cover* of \mathcal{C} . \square

We end this section with a combinatorial construction. Let \mathcal{C} be a given finite EI category. Then we associate to it an *underlying quiver* $Q = (Q_0, Q_1)$ and an *underlying poset* (Q_0, \leq) . The set of vertices $Q_0 = \text{Ob } \mathcal{C}$. The set of arrows Q_1 is defined in the following way. By the first statement of Proposition 2.3, the set of unfactorizable morphisms from an object x to another object y is closed under the actions of $\mathcal{C}(x, x)$ and $\mathcal{C}(y, y)$. Choose a fixed representative for each $(\mathcal{C}(y, y), \mathcal{C}(x, x))$ -orbit. Repeating this process for all pairs of different objects $x \neq y$, we get a set $A = \{\alpha_1, \dots, \alpha_n\}$ of orbit representatives. Elements in A are called *representative unfactorizable morphisms*. Then we put an arrow $x \rightarrow y$ in Q_1 for each representative unfactorizable morphism $\alpha : x \rightarrow y$ in A . The reader can check that Q is acyclic since \mathcal{C} is skeletal.

The partial order \leq is defined as follows. We let $x \leq y$ if and only if there is a morphism $\alpha : x \rightarrow y$ in \mathcal{C} . The reader can check that \leq defined in this way is indeed a partial order (keep in mind that \mathcal{C} is skeletal). Moreover, it is easy to see that \leq can also be defined by considering the underlying quiver $Q = (Q_0, Q_1)$. That is, $x \leq y$ if and only if there is an oriented path from x to y in Q .

3. ELEMENTARY REDUCTIONS

From now on we begin to consider the representation types of finite EI categories. We describe some easy but useful techniques and results in this section.

Let $F : \mathcal{D} \rightarrow \mathcal{C}$ be a functor between two finite EI categories. Then F induces a functor $\text{Res}_F : \mathcal{C}\text{-rep} \rightarrow \mathcal{D}\text{-rep}$. Explicitly, for $M \in \mathcal{C}\text{-rep}$, $\text{Res}_F(M) = M \circ F$ is a functor from \mathcal{D} to $k\text{-vec}$. This functor Res_F is called the *restriction along F* (Definition 3.2.1 in [24]). In the case that F is injective on $\text{Ob } \mathcal{D}$, it induces an algebra homomorphism $\varphi_F : k\mathcal{D} \rightarrow k\mathcal{C}$ by sending $\alpha \in \text{Mor } \mathcal{D}$ to $F(\alpha) \in \text{Mor } \mathcal{C}$ (Proposition 3.1, [22]). In that case, the restriction functor Res_F is precisely the usual restriction functor from $k\mathcal{C}\text{-mod}$ to $k\mathcal{D}\text{-mod}$ induced by the algebra homomorphism φ_F .

An important case is that \mathcal{D} is a subcategory of \mathcal{C} and F is the inclusion functor. Then $k\mathcal{D}$ is a subalgebra of $k\mathcal{C}$ and φ_F is the inclusion. In this case we define the *induction functor* $\uparrow_{\mathcal{D}}^{\mathcal{C}}$ mapping $N \in k\mathcal{D}\text{-mod}$ to $k\mathcal{C} \otimes_{k\mathcal{D}} N \in k\mathcal{C}\text{-mod}$. The *co-induction functor* $\uparrow_{\mathcal{D}}^{\mathcal{C}}$ sends N to $N \uparrow_{\mathcal{D}}^{\mathcal{C}} = \text{Hom}_{k\mathcal{C}}(k\mathcal{C}, N)$.

In his thesis Xu has developed an induction-restriction theory for finite EI categories. We do not repeat his results here and suggest the reader to look at [24, 25] for a detailed description. Although many results in the representation theory of

finite groups generalize to finite EI categories, there exist differences, as shown by the following example.

Example 3.1. Let \mathcal{C} be the following EI category such that g has order 2 and permutes α and β . This category has finite representation type.

$$\mathcal{C} : \quad \textcirclearrowleft x \begin{matrix} \xrightarrow{\alpha} \\ \xrightarrow{\beta} \end{matrix} y \textcirclearrowright \langle g \rangle$$

Let \mathcal{D} be the subcategory formed by deleting the morphism g . Actually, \mathcal{D} viewed as a quiver is the Kronecker quiver. Therefore, it is of infinite representation type.

Suppose that $\text{char}(k) = 0$. Then $R : k \xrightarrow[3]{1} k$ is a representation of \mathcal{D} . Clearly, $\dim_k R = 2$. Let $\{u, v\}$ be a basis of R such that $\alpha \cdot u = v$ and $\beta \cdot u = 3v$.

Consider the induced module $\tilde{R} : k\mathcal{C} \otimes_{k\mathcal{D}} R$. We have

$$g \otimes v = g \otimes \alpha \cdot u = g\alpha \otimes u = \beta \otimes u = 1_y \otimes \beta \cdot u = 1_y \otimes 3v = 3(1_y \otimes v).$$

Therefore,

$$1_y \otimes v = g^2 \otimes v = g(g \otimes v) = g(3(1_y \otimes v)) = 3(g \otimes v) = 9(1_y \otimes v).$$

Since $\text{char}(k) = 0$, we get $1_y \otimes v = 0$. Consequently,

$$\alpha \otimes u = 1_y \otimes \alpha \cdot u = 1_y \otimes v = 0$$

and similarly $\beta \otimes u = 0$. Therefore, \tilde{R} is spanned by $1_x \otimes u$, so is one-dimensional! Actually, it has the form $\tilde{R}(x) \cong k \rightarrow \tilde{R}(y) = 0$.

The following result relates the representation type of \mathcal{C} to that of a subcategory.

Proposition 3.2. Let \mathcal{D} be a subcategory of a finite EI category \mathcal{C} .

- (1) If $M \mid M \downarrow_{\mathcal{D}}^{\mathcal{C}} \uparrow_{\mathcal{D}}^{\mathcal{C}}$ for all $M \in k\mathcal{C}\text{-mod}$, then \mathcal{C} is of finite representation type whenever so is \mathcal{D} .
- (2) Dually, if $N \mid N \uparrow_{\mathcal{D}}^{\mathcal{C}} \downarrow_{\mathcal{D}}^{\mathcal{C}}$ for all $N \in k\mathcal{D}\text{-mod}$, then \mathcal{C} is of infinite representation type whenever so is \mathcal{D} .
- (3) If $k\mathcal{D} \mid ({}_k\mathcal{D} k\mathcal{C}_{k\mathcal{D}})$, then $N \mid N \uparrow_{\mathcal{D}}^{\mathcal{C}} \downarrow_{\mathcal{D}}^{\mathcal{C}}$ for every $N \in k\mathcal{D}\text{-mod}$. In particular, if \mathcal{D} is of infinite representation type, so is \mathcal{C} .

Proof. By the given assumption, we know that for every indecomposable $M \in k\mathcal{C}\text{-mod}$, there exists some indecomposable $N \in k\mathcal{D}\text{-mod}$ such that $M \mid N \uparrow_{\mathcal{D}}^{\mathcal{C}}$. If there are only finitely many indecomposable $k\mathcal{D}$ -modules (up to isomorphism), then induced to \mathcal{C} we can only get finitely many indecomposable $k\mathcal{C}$ -modules. Therefore, \mathcal{C} must be of finite representation type. This proves the first statement. The second one can be shown in a similar way.

Let $N \in k\mathcal{D}\text{-mod}$. Then

$$\begin{aligned} N \uparrow_{\mathcal{D}}^{\mathcal{C}} \downarrow_{\mathcal{D}}^{\mathcal{C}} &= {}_{k\mathcal{D}}(k\mathcal{C} \otimes_{k\mathcal{D}} N) = ({}_k\mathcal{D} k\mathcal{C}_{k\mathcal{D}}) \otimes_{k\mathcal{D}} N \\ &\cong ({}_k\mathcal{D} \otimes_{k\mathcal{D}} N) \oplus (R \otimes_{k\mathcal{D}} N) \cong N \oplus (R \otimes_{k\mathcal{D}} N), \end{aligned}$$

Where R is a $(k\mathcal{D}, k\mathcal{D})$ -bimodule. Thus N is a direct summand of $N \uparrow_{\mathcal{D}}^{\mathcal{C}} \downarrow_{\mathcal{D}}^{\mathcal{C}}$. In particular, if \mathcal{D} is of infinite representation type, so is \mathcal{C} by (2). \square

If furthermore \mathcal{D} is a full subcategory of \mathcal{C} , then $k\mathcal{D} = 1_{\mathcal{D}} k\mathcal{C} 1_{\mathcal{D}}$, and the restriction functor Res_F sends $M \in k\mathcal{C}\text{-mod}$ to $M \downarrow_{\mathcal{D}}^{\mathcal{C}} = 1_{\mathcal{D}} M$. It has a left adjoint functor $\uparrow_{\mathcal{D}}^{\mathcal{C}}$ and a right adjoint functor $\downarrow_{\mathcal{D}}^{\mathcal{C}}$.

Lemma 3.3. *Let \mathcal{D} be a full subcategory of a finite EI category \mathcal{C} and $N \in k\mathcal{D}\text{-mod}$. If S is a subset of $\text{Ob } \mathcal{D}$ on which N is generated by its values, then the induced module $N \uparrow_{\mathcal{D}}^{\mathcal{C}} = k\mathcal{C} \otimes_{k\mathcal{D}} N$ is also generated by its values on S . Furthermore, $N \uparrow_{\mathcal{D}}^{\mathcal{C}} \downarrow_{\mathcal{D}}^{\mathcal{C}} \cong N$.*

Proof. If S is a subset of $\text{Ob } \mathcal{D}$ on which N is generated by its values, we have $N = k\mathcal{D} \cdot \bigoplus_{x \in S} N(x)$, then:

$$N \uparrow_{\mathcal{D}}^{\mathcal{C}} = k\mathcal{C} \otimes_{k\mathcal{D}} N = k\mathcal{C} \otimes_{k\mathcal{D}} (k\mathcal{D} \cdot \bigoplus_{x \in S} N(x)) = k\mathcal{C} \cdot \bigoplus_{x \in S} (1_{\mathcal{D}} \otimes_{k\mathcal{D}} N(x)),$$

which is also generated by its values on S .

The second statement is true because:

$$N \uparrow_{\mathcal{D}}^{\mathcal{C}} \downarrow_{\mathcal{D}}^{\mathcal{C}} = 1_{\mathcal{D}}(k\mathcal{C} \otimes_{k\mathcal{D}} N) = 1_{\mathcal{D}} k\mathcal{C} 1_{\mathcal{D}} \otimes_{k\mathcal{D}} N = k\mathcal{D} \otimes_{k\mathcal{D}} N \cong N.$$

This finishes the proof. \square

We get an immediate corollary.

Proposition 3.4. *Let \mathcal{D} be a full subcategory of a finite EI category \mathcal{C} . If \mathcal{D} is of infinite representation type, so is \mathcal{C} .*

Proof. Follows from Proposition 3.2 and the previous lemma. \square

Now we consider another type of important functors $G : \mathcal{C} \rightarrow \mathcal{D}$ between two finite EI categories, called *quotient functors*. That is, G is a full functor and is bijective restricted to objects. Correspondingly, we call \mathcal{D} a *quotient category* of \mathcal{C} . By Proposition 3.1 in [22], G induces an algebra homomorphism $\varphi_G : k\mathcal{C} \rightarrow k\mathcal{D}$. Moreover, this homomorphism is surjective, so $k\mathcal{D}$ is a quotient algebra of $k\mathcal{C}$, and hence its infinite representation type implies the infinite representation type of $k\mathcal{C}$. Namely we proved

Proposition 3.5. *Let \mathcal{D} be a quotient category of a finite EI category \mathcal{C} . If \mathcal{C} is of finite representation type, so is \mathcal{D} .*

For instance, \mathcal{C} is a quotient category of its free EI cover $\hat{\mathcal{C}}$. In this case we have

Proposition 3.6. *Let \mathcal{C} be a finite EI category and $\hat{\mathcal{C}}$ be its free EI cover. Then the category algebra $k\mathcal{C}$ is a quotient algebra of $k\hat{\mathcal{C}}$. If $k\mathcal{C} \cong k\hat{\mathcal{C}}/I$, then the $k\hat{\mathcal{C}}$ -ideal I as a vector space is spanned by elements of the form $\hat{\alpha} - \hat{\beta}$, where $\hat{\alpha}$ and $\hat{\beta}$ are morphisms in $\hat{\mathcal{C}}$ for which the images in \mathcal{C} coincide.*

Proof. See Lemma 6.1 in [15]. \square

We obtain the following elementary criterion for finite representation type.

Proposition 3.7. *Let \mathcal{C} be a finite EI category of finite representation type. Then the following conditions must be true:*

- (1) *The group $\mathcal{C}(x, x)$ is of finite representation type for each $x \in \text{Ob } \mathcal{C}$.*
- (2) *The underlying poset \mathcal{P} of \mathcal{C} is of finite representation type.*
- (3) *For $x \neq y \in \text{Ob } \mathcal{C}$, $\mathcal{C}(x, y)$ has at most one orbit as a $(\mathcal{C}(y, y), \mathcal{C}(x, x))$ -biset.*
- (4) *If $x, y, z \in \text{Ob } (\mathcal{C})$ are distinct such that $\mathcal{C}(x, y)$ and $\mathcal{C}(y, z)$ are non-empty, then $\mathcal{C}(x, z) = \mathcal{C}(y, z) \circ \mathcal{C}(x, y) = \{\beta \circ \alpha \mid \alpha \in \mathcal{C}(x, y), \beta \in \mathcal{C}(y, z)\}$.*

Proof. (1): Apply Proposition 3.4 to the full subcategory formed by x .

(2): There is a quotient functor $F : \mathcal{C} \rightarrow \mathcal{P}$. It is the identity map restricted to $\text{Ob } \mathcal{C}$, and sends every morphism (if exists) in $\mathcal{C}(x, y)$ to the unique morphism $\mathcal{P}(x, y)$ for $x, y \in \text{Ob } \mathcal{C}$. The conclusion follows from Proposition 3.5.

(3): Consider the full subcategory \mathcal{D} with objects x and y . Let O_1, O_2, \dots, O_n be the $(\mathcal{D}(y, y), \mathcal{D}(x, x))$ -orbits of $\mathcal{D}(x, y) = \mathcal{C}(x, y)$. Take a representative α_i for each orbit O_i , $1 \leq i \leq n$. Let \mathcal{E} be the following category: $\text{Ob } \mathcal{E} = \{x, y\}$ and $\text{Mor } \mathcal{E} = \{1_x, 1_y, \alpha_1, \dots, \alpha_n\}$. We claim that \mathcal{E} is a quotient category of \mathcal{D} .

Indeed, define $G : \mathcal{D} \rightarrow \mathcal{E}$ as follows. Restricted to $\text{Ob } \mathcal{D}$, G is the identity map; $G(\delta) = 1_x$ for $\delta \in \mathcal{D}(x, x)$; $G(\rho) = 1_y$ for $\rho \in \mathcal{D}(y, y)$; and $G(\alpha) = \alpha_i$ for $\alpha \in O_i \subseteq \mathcal{D}(x, y)$, $1 \leq i \leq n$. We check that G is well defined. Therefore \mathcal{E} is a quotient category of \mathcal{D} , so it must be of finite representation type. But this happens if and only if $n \leq 1$.

(4): Clearly, $\mathcal{C}(y, z) \circ \mathcal{C}(x, y) \subseteq \mathcal{C}(x, z)$. We show the other inclusion by contradiction. Suppose that $\mathcal{C}(y, z) \circ \mathcal{C}(x, y)$ is a proper subset of $\mathcal{C}(x, z)$. We can take $\varphi, \psi \in \mathcal{C}(x, z)$ such that $\varphi \in \mathcal{C}(y, z) \circ \mathcal{C}(x, y)$ and $\psi \notin \mathcal{C}(y, z) \circ \mathcal{C}(x, y)$. Then the $(\mathcal{C}(z, z), \mathcal{C}(x, x))$ -orbit generated by φ is contained in $\mathcal{C}(y, z) \circ \mathcal{C}(x, y)$. So φ and ψ are in different orbits. Therefore, $\mathcal{C}(x, z)$ as a $(\mathcal{C}(z, z), \mathcal{C}(x, x))$ -biset has more than one orbits. By (3), \mathcal{C} is of infinite representation type. The conclusion follows from contradiction. \square

Let $p = \text{chak}(k)$, which is 0 or a prime. It is well known that for a finite group G , the group algebra kG is of finite representation type if and only if it has a cyclic Sylow p -subgroup (when $p = 0$ the trivial group $1 \leq G$ is the unique Sylow 0-subgroup by our convention). Therefore, if \mathcal{C} is of finite representation type, then the automorphism group of every object in \mathcal{C} has cyclic Sylow p -subgroups.

By the last two statements in the above proposition, if \mathcal{C} is of finite representation type, then every nonempty $\mathcal{C}(x, y)$ is generated by a unique element α_{xy} as a $(\mathcal{C}(y, y), \mathcal{C}(x, x))$ -biset. Furthermore, these generators can be chosen to satisfy $\alpha_{yz} \circ \alpha_{xy} = \alpha_{xz}$.

4. ALL AUTOMORPHISM GROUPS HAVE INVERTIBLE ORDERS

From now on we only consider finite EI categories \mathcal{C} satisfying conditions (1-4) in Proposition 3.7 since otherwise we can immediately conclude that \mathcal{C} is of infinite representation type. Throughout this section we assume that the automorphism groups of all objects in \mathcal{C} have invertible orders. Therefore, the group algebra $k\mathcal{C}(x, x)$ is semisimple for every $x \in \text{Ob } \mathcal{C}$. With this assumption, we can construct the ordinary quiver \tilde{Q} for $k\mathcal{C}$. Clearly, $k\tilde{Q}$ provide some information on the representation type of \mathcal{C} .

The content of this section comes from [14]. The reader is suggested to refer to that paper for details.

We introduce some notation here. Let $\alpha : x \rightarrow y$ be a non-isomorphism in a finite EI category \mathcal{C} . Let G and H be $\mathcal{C}(x, x)$ and $\mathcal{C}(y, y)$ respectively. Define: $G_0 = \text{Stab}_G(\alpha) = \{g \in G \mid \alpha g = \alpha\}$, $H_0 = \text{Stab}_H(\alpha) = \{h \in H \mid h\alpha = \alpha\}$, $G_1 = \text{Stab}_G(H\alpha) = \{g \in G \mid H\alpha g = H\alpha\} = \{g \in G \mid \exists h \in H \text{ with } \alpha g = h\alpha\}$, and $H_1 = \text{Stab}_H(\alpha G) = \{h \in H \mid h\alpha G = \alpha G\} = \{h \in H \mid \exists g \in G \text{ with } h\alpha = \alpha g\}$.

Lemma 4.1. *With the above notation, $G_0 \triangleleft G_1 \leq G$ and $H_0 \triangleleft H_1 \leq H$. Moreover, $G_1/G_0 \cong H_1/H_0$.*

Proof. This is Lemma 3.3 in [14], where we do not use the assumption that α is unfactorizable. \square

By this lemma, we identify the permutation module $k \uparrow_{G_0}^{G_1}$ with $k \uparrow_{H_0}^{H_1}$.

Now we construct the ordinary quiver \tilde{Q} for \mathcal{C} . The detailed algorithm is as follows:

Step 1: The vertex set of \tilde{Q} is $\bigsqcup_{x \in \text{Ob } \mathcal{C}} S_x$, where S_x is a set of representatives of the isomorphism classes of simple $k\mathcal{C}(x, x)$ -modules.

Step 2: Let $\alpha : x \rightarrow y$ be a representative unfactorizable morphism. Then it determines uniquely:

- $G = \mathcal{C}(x, x)$, $G_0 = \text{Stab}_G(\alpha)$, $G_1 = \text{Stab}_G(H\alpha)$;
- $H = \mathcal{C}(y, y)$, $H_0 = \text{Stab}_H(\alpha)$, $H_1 = \text{Stab}_H(\alpha G)$;
- $\{V_1, \dots, V_m\}$: the set of pairwise non-isomorphic simple kG -modules;
- $\{W_1, \dots, W_n\}$: the set of pairwise non-isomorphic simple kH -modules;
- $\{U_1, \dots, U_r\}$: the set of pairwise non-isomorphic simple summands of $k \uparrow_{G_0}^{G_1}$.

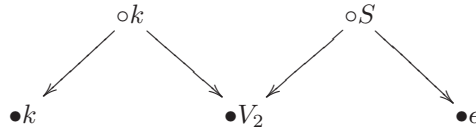
Step 3: For each particular simple kG -module V in $\{V_1, \dots, V_m\}$ choose a decomposition $V \downarrow_{G_1}^G \cong U_1^{e_1} \oplus \dots \oplus U_r^{e_r} \oplus X$ where X has no summand isomorphic to any U_i . For each simple kH -module W in $\{W_1, \dots, W_n\}$ choose a decomposition $W \downarrow_{H_1}^H \cong U_1^{f_1} \oplus \dots \oplus U_r^{f_r} \oplus Y$ such that Y has no summand isomorphic to any U_i . Then we put $\sum_{i=1}^r e_i f_i$ arrows from the vertex V to the vertex W in \tilde{Q} .

Step 4: Repeat Steps 2-4 for all representative unfactorizable morphisms.

The following simple example (Example 4.4 in [14]) illustrates our construction.

Example 4.2. Let \mathcal{C} be a finite EI category with objects x and y ; $H = \mathcal{C}(y, y)$ is a copy of the symmetric group S_3 on 3 letters; $G = \mathcal{C}(x, x)$ is cyclic of order 2; $\mathcal{C}(x, y) = S_3$ regarded as an (H, G) -biset where H acts from the left by multiplication, G acts from the right by multiplication after identifying G with a subgroup G^\dagger of S_3 ; $\mathcal{C}(y, x) = \emptyset$. Let k be the complex field.

Since $kG \cong k \oplus S$, and $kH \cong k \oplus \epsilon \oplus V_2 \oplus V_2$, the ordinary quiver \tilde{Q} has 5 vertices: $\circ k$ and $\circ S$ corresponding to x ; $\bullet k$, $\bullet V_2$ and $\bullet \epsilon$ corresponding to y . We choose $\alpha = 1 \in S_3$ as the representative unfactorizable morphism and then $G_0 = 1$, $G_1 = G$, $H_0 = 1$, $H_1 = G^\dagger$. Therefore $k \downarrow_{H_1}^H \cong k$, $\epsilon \downarrow_{H_1}^H \cong S$, $V_2 \downarrow_{H_1}^H \cong S \oplus k$. Thus the ordinary quiver \tilde{Q} is as follows:



The following proposition is Proposition 4.5 in [14].

Proposition 4.3. *Let \mathcal{C} be a finite EI category for which the automorphism groups of all objects have invertible orders. Then the ordinary quiver \tilde{Q} of $k\mathcal{C}$ is acyclic and contains the underlying quiver Q of \mathcal{C} as a subquiver.*

If furthermore \mathcal{C} is a finite free EI category, its representation type can be determined as follows:

Corollary 4.4. *Let \mathcal{C} be a finite free EI category for which the automorphism groups of all objects have invertible orders. Then \mathcal{C} is of finite representation type if and only if the ordinary quiver \tilde{Q} of $k\mathcal{C}$ is a disjoint union of Dynkin quivers.*

Proof. By (3) of Proposition 2.6, $k\mathcal{C}$ is hereditary. Therefore, $k\mathcal{C}$ is Morita equivalent to $k\tilde{Q}$. \square

5. ALL AUTOMORPHISM GROUPS ARE p -GROUPS

In this section we consider a family of special finite EI categories \mathcal{C} , i.e., the automorphism groups of all objects in \mathcal{C} are p -groups, where $\text{char}(k) = p > 0$. If \mathcal{C} has only two objects, its representation type can be determined by Bongartz's list in [3, 5]. This work is described in [11]. We record his result here.

Proposition 5.1. *(Theorem 5.3 in [11]) Let \mathcal{C} be a finite EI category with two objects for which the automorphism groups are p -groups. Suppose that $\mathcal{C}(y, x) = \emptyset$. Let $G = \mathcal{C}(x, x)$ and $H = \mathcal{C}(y, y)$. Then \mathcal{C} is of finite representation type if and only if the following conditions are satisfied*

- (1) both G and H are cyclic;
- (2) either G or H acts transitively on $\mathcal{C}(x, y)$;
- (3) one of the following is true:
 - (a) $|\mathcal{C}(x, y)| \leq 1$;
 - (b) $|G||H| \leq 3$;
 - (c) $p = 2 = \mathcal{C}(x, y)$, either G or H is trivial;
 - (d) $p = 2 = \mathcal{C}(x, y)$, either G or H has order 2 and acts transitively;
 - (e) $p = 3 = |\mathcal{C}(x, y)| = |G| = |H|$, and both G and H act transitively.

An immediate corollary is:

Corollary 5.2. *Suppose that $p \geq 5$. Let \mathcal{C} be a skeletal finite EI category for which the automorphism groups of all objects are p -groups. If \mathcal{C} is of finite representation type, then $|\mathcal{C}(x, y)| \leq 1$ for every pair of distinct objects x and y .*

Proof. Consider the full subcategory with objects x and y . This subcategory should be of finite representation type as well by Proposition 3.4. Applying the above proposition the conclusion follows. \square

Now we classify the representation types of finite EI categories with three objects for which all automorphism groups are p -groups, where $p \geq 5$. The main techniques we use are covering theory (see [5]) and *string algebras* (see [8]). By the above corollary, we can make the following assumptions: all automorphism groups are cyclic, and there is at most one morphism between every pair of distinct objects.

Consider two families of finite EI categories \mathcal{C} as below:



Let $|G| = p^r$, $|H| = p^s$, and $|L| = p^t$. Note that the category algebra $k\mathcal{C}$ is isomorphic to the path algebra of one of the following bounded quivers with relations $g^{p^r} = h^{p^s} = l^{p^t} = 0$, $h\alpha = \alpha g = 0$, and $l\beta = \beta h = 0$ (or $\beta l = h\beta = 0$ for the second class).



Proposition 5.3. *Let \mathcal{C} be a connected skeletal finite EI category with three objects such that the automorphism groups of all objects are p -groups and suppose that $\text{char}(k) = p \geq 5$. Then \mathcal{C} is of finite representation type if and only if \mathcal{C} or \mathcal{C}^{op} satisfies one of the following conditions:*

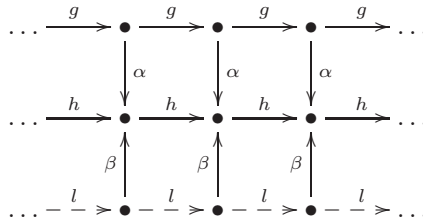
- (1) *it lies in the first family;*
- (2) *it lies in the second family and $s = 0$;*
- (3) *it lies in the second family and $r = t = 0$.*

Proof. If \mathcal{C} is of finite representation type, then either \mathcal{C} or \mathcal{C}^{op} lies in one of these two families. Without loss of generality we suppose that \mathcal{C} are in one of these two families. There are four cases:

Case I: If \mathcal{C} is in the first family, then $k\mathcal{C}$ is a *string algebra*. For a definition of string algebras, see [8]. In that paper an algorithm to construct all indecomposable representations (up to isomorphism) of a string algebra is described. Thus we can describe all indecomposable representations R of \mathcal{C} as follows: for $w \in \{x, y, z\}$, $R(w)$ is an indecomposable representation of the automorphism group of the corresponding object; the maps $R(\alpha)$ and $R(\beta)$ send the top (if nonzero) of a group representation isomorphically onto the socle (if nonzero) of another group representation. Obviously, \mathcal{C} is of finite representation type.

Case II: \mathcal{C} lies in the second family, and $s = 0$. In this case $k\mathcal{C}$ is still a string algebra, and we can describe all indecomposable representations R of \mathcal{C} as above. Similarly, the value of R on each object is an indecomposable representation of the automorphism group of this object, so \mathcal{C} is of finite representation type as well.

Case III: \mathcal{C} lies in the second family, $s \neq 0$, and either $r \neq 0$ or $t \neq 0$. Without loss of generality we assume that $r \neq 0$. Then $k\mathcal{C}$ has a covering (see [5]) as shown below with relations $g^{p^r} = h^{p^s} = l^{p^t} = 0$, $\alpha g = h\alpha = 0$, and $h\beta = \beta l = 0$. Here the arrow marked by l does not exist if $t = 0$.



This quiver is not locally representation-finite since it has the following subquiver with $2p + 1 \geq 11$ vertices, which is of infinite representation type:

$$\begin{array}{ccccccccccc}
 1 & \xrightarrow{h} & 2 & \xrightarrow{h} & \dots & \xrightarrow{h} & p & \xleftarrow{\alpha} & p+1 & \xrightarrow{g} & p+2 & \xrightarrow{g} & \dots & \xrightarrow{g} & 2p \\
 & & & & & & \uparrow \beta & & & & & & & & \\
 & & & & & & 0 & & & & & & & &
 \end{array}$$

Therefore, \mathcal{C} is of infinite representation type.

Case IV: \mathcal{C} lies in the second family, and $r = t = 0$. We claim that this category is of finite representation type. To prove this, let R be an indecomposable representation of \mathcal{C} . It suffices to show that $R(y)$ is an indecomposable kH -module. Otherwise, $R(y) = M \oplus N$ where $M, N \in kH\text{-mod}$ are nonzero and M is indecomposable. Note that $R(\alpha)$ and $R(\beta)$ are injective, and their images are contained in $\text{Soc}(R(y))$. Take $0 \neq v \in \text{Soc}(M)$, and let u and w be the preimages of v in $R(x)$ and $R(z)$ respectively (if a preimage does not exist, we take 0 alternately). Then the reader can check that the $k\mathcal{C}$ -submodule R' of R with $R'(x) = \langle u \rangle$, $R'(y) = M$ and $R'(z) = \langle w \rangle$ is a nonzero proper summand of R . This contradicts the assumption that R is indecomposable. Therefore, $R(y)$ is indecomposable, so \mathcal{C} is of finite representation type. \square

Remark 5.4. We proved the finite representation type for \mathcal{C} with three objects which lies in the first family. Actually, the proof works for finite EI categories with arbitrarily many objects such that all morphisms have the same direction.

Two quivers with isomorphic underlying graphs have the same representation type, i.e., the representation type of a quiver is independent of the orientations of arrows. This is no longer true for finite EI categories, as shown by the above proposition.

6. FINITE EI CATEGORIES WITH TWO OBJECTS

Throughout this section let \mathcal{C} be a (connected, skeletal) finite EI category with two objects x and y . Without loss of generality we assume that $\mathcal{C}(x, y) \neq \emptyset$, but $\mathcal{C}(y, x) = \emptyset$. Conditions (2) and (4) in Proposition 3.7 are satisfied trivially. We suppose that conditions (1) and (3) also hold. That is, both $\mathcal{C}(x, x)$ and $\mathcal{C}(y, y)$ have cyclic Sylow p -subgroups, and $\mathcal{C}(x, y)$ has only one orbit as a biset.

Let $G = \mathcal{C}(x, x)$ and $H = \mathcal{C}(y, y)$. Fix a representative morphism $\alpha : x \rightarrow y$ in the unique orbit. Then the structure of \mathcal{C} can be pictured as below:

$$\mathcal{C} : \quad \begin{array}{ccc} \curvearrowright_G x & \xrightarrow{H\alpha G} & y \curvearrowright_H \end{array}$$

The following notation will be used in this section: $G_0 = \text{Stab}_G(\alpha)$, $H_0 = \text{Stab}_H(\alpha)$, $G_1 = \text{Stab}_G(H\alpha)$, and $H_1 = \text{Stab}_H(\alpha G)$. By Lemma 4.1, $G_0 \triangleleft G_1 \leq G$, $H_0 \triangleleft H_1 \leq H$, and $G_1/G_0 \cong H_1/H_0$. We identify these two quotient groups. Clearly, G acts transitively on $\mathcal{C}(x, y)$ if and only if $H_1 = H$, and H acts transitively on $\mathcal{C}(x, y)$ if and only if $G_1 = G$.

6.1. A description of representations of \mathcal{C} . In this subsection we describe representations R of \mathcal{C} . It is easy to check that $R(x)$ (resp., $R(y)$) is a kG -module (resp., a kH -module). Moreover, suppose that $(\varphi_x, \varphi_y) : R_1 \rightarrow R_2$ is a $k\mathcal{C}$ -module homomorphism, then $\varphi_x : R_1(x) \rightarrow R_2(x)$ is a kG -module homomorphism, and

$\varphi_y : R_1(y) \rightarrow R_2(y)$ is a kH -module homomorphism. For an arbitrary $\beta \in \mathcal{C}(x, y)$, since $\mathcal{C}(x, y)$ has only one orbit as a biset, we can find some $g \in G$ and $h \in H$ such that $\beta = h\alpha g$. Therefore, $R(\beta) = R(h)R(\alpha)R(g)$.

From the above analysis we know that R is uniquely determined by a linear map $R(\alpha)$ from a kG -module V to a kH -module W . Clearly, not every linear map $\varphi : V \rightarrow W$ can be used to define a representation of \mathcal{C} . We characterize in the next proposition those linear maps which indeed define representations of \mathcal{C} . Let V^\perp be the kernel of φ , $\bar{V} = V/V^\perp$, and W^\top be the image of φ . The map φ gives rise to a vector space isomorphism $\bar{\varphi} : \bar{V} \rightarrow W^\top$ defined by $\bar{\varphi}(\bar{v}) = \varphi(v)$.

Proposition 6.1. *Notation as above. The linear map $\varphi : V \rightarrow W$ determines a representation R of \mathcal{C} by defining $R(\alpha) = \varphi$ if and only if \bar{V} (resp., W^\top) is a kG_1 -module (resp., a kH_1 -module) on which G_0 (resp., H_0) acts trivially, and $\bar{\varphi}$ is a $k(G_1/G_0) \cong k(H_1/H_0)$ -module isomorphism.*

Proof. Suppose that there exists a representation R of \mathcal{C} such that $R(\alpha) = \varphi$, i.e., $\varphi(v) = \alpha v$ for all $v \in V$. Take $w \in W^\top$. Then we can find some $v \in V$ such that $\alpha v = w$. For every $h \in H_0$, we get $hw = h\alpha v = \alpha v = w$. Thus H_0 acts trivially on W^\top . Moreover, for $h \in H_1$, we can find some $g \in G$ such that $h\alpha = \alpha g$. Therefore, $hw = h\alpha v = \alpha gv = \varphi(gv)$, so $hw \in W^\top$. Consequently, W^\top is a kH_1 -module.

Now consider \bar{V} . First, for every $g \in G_1$ and $v \in V^\perp$, we can find some $h \in H$ such that $h\alpha = \alpha g$. Therefore, $\alpha(gv) = (h\alpha)v = h(\alpha v) = h\varphi(v) = 0$ since v is in the kernel of φ . We deduce that V^\perp is a kG_1 -module, so \bar{V} is a kG_1 -module as well. For every $g \in G_0$ and $\bar{v} \in \bar{V}$, we have $\alpha(gv - v) = \alpha gv - \alpha v = \alpha v - \alpha v = 0$. Therefore, $gv - v \in V^\perp$, and $g\bar{v} - \bar{v} = 0$. Thus G_0 acts trivially on \bar{V} .

Recall that the isomorphism $\rho : G_1/G_0 \rightarrow H_1/H_0$ is defined in the following way (see the proof of Lemma 3.3 in [14]): For a given $g \in G_1$, we can find some $h \in H_1$ such that $\alpha g = h\alpha$. Then ρ sends \bar{g} to \bar{h} . Now the reader can check that this isomorphism gives an isomorphism between \bar{V} and W^\top as $k(G_1/G_0) \cong k(H_1/H_0)$ -modules. This proves the only if part.

Conversely, if φ satisfies the given conditions, then by defining $R(\alpha)(v) = \bar{\varphi}(\bar{v})$ we indeed obtain a representation R of \mathcal{C} . To prove this, it suffices to show that if $h_1\alpha g_1 = h_2\alpha g_2$ for $g_1, g_2 \in G$ and $h_1, h_2 \in H$, then $h_1\bar{\varphi}(\bar{g_1 v}) = h_2\bar{\varphi}(\bar{g_2 v})$ for every $v \in V$. That is, $h_2^{-1}h_1\bar{\varphi}(\bar{g_1 v}) = \bar{\varphi}(\bar{g_2 v})$, where $\bar{g_1 v}, \bar{g_2 v}$ are the images of $g_1 v$ and $g_2 v$ in \bar{V} respectively.

Note that $h_1\alpha g_1 = h_2\alpha g_2$ implies $h_2^{-1}h_1\alpha = \alpha g_2 g_1^{-1}$, so $h_2^{-1}h_1$ and $g_2 g_1^{-1}$ are contained in H_1 and G_1 respectively. Passing to the quotient groups, the isomorphism ρ sends $g_2 g_1^{-1} \in G_1/G_0$ to $h_2^{-1}h_1 \in H_1/H_0$. Since \bar{V} is a kG_1 -module on which G_0 acts trivially and W^\top is a kH_1 -module on which H_0 acts trivially, and $\bar{\varphi}$ is a $k(G_1/G_0) \cong k(H_1/H_0)$ -module isomorphism, we get

$$h_2^{-1}h_1\bar{\varphi}(\bar{g_1 v}) = \overline{h_2^{-1}h_1\bar{\varphi}(\bar{g_1 v})} = \bar{\varphi}(\overline{g_2 g_1^{-1}} \cdot \bar{g_1 v}).$$

Note that in the above identity $\bar{g_1 v}$ cannot be expressed as $g_1 \bar{v}$ since \bar{V} is only a kG_1 -module instead of a kG -module, and the action of $g_1 \in G$ on \bar{v} in general is not defined. However, we have

$$\overline{g_1 v} = \overline{g_1 g_2^{-1} g_2 v} = (g_1 g_2^{-1}) \cdot \overline{g_2 v} = \overline{g_1 g_2^{-1}} \cdot \bar{g_2 v}$$

since \bar{V} is a kG_1 -module, $g_1g_2^{-1} \in G_1$, and G_0 acts trivially on \bar{V} . Combining these two identities together, we get

$$h_2^{-1}h_1\bar{\varphi}(\overline{g_1v}) = \bar{\varphi}(\overline{g_2g_1^{-1}} \cdot \overline{g_1v}) = \bar{\varphi}(\overline{g_2g_1^{-1}g_1g_2^{-1}} \cdot \overline{g_2v}) = \bar{\varphi}(\overline{g_2v})$$

as required. This finishes the proof. \square

As an example, let us construct a representation for the finite EI category introduced in Example 4.2.

Example 6.2. Let \mathcal{C} be the finite EI category described in Example 4.2. Recall $H = S_3$ is the symmetric group on three letters; $G = C_2$ is a cyclic group of order 2; $\mathcal{C}(x, y) \cong S_3$ regarded as an (H, G) -biset where H acts from the left by multiplication and G acts from the right by multiplication after identifying G with a subgroup $H_1 \leq S_3$. We have $G_0 = 1$, $G_1 = G$, $H_0 = 1$, $H_1 \cong C_2$. We set $\text{char}(k) = 3$ instead of 0 as before.

Let $V = kG$ and W be the projective kH -module with $W/\text{rad } W \cong k$. Then

$$V = V \downarrow_{G_1}^G = k \uparrow_{G_0}^G \cong k \oplus \epsilon, \quad W \downarrow_{H_1}^H \cong k \oplus k \oplus \epsilon, \quad k \uparrow_{H_0}^{H_1} \cong k \oplus \epsilon.$$

Therefore, any linear map φ with the following matrix representation can define a representation of \mathcal{C} :

$$\begin{bmatrix} \lambda & 0 \\ \mu & 0 \\ 0 & \nu \end{bmatrix}, \quad \lambda, \mu, \nu \in k.$$

6.2. Biset decomposition and induction. In Section 3 we have defined the restriction functor and induction functor. Now we use these functors to study some special subcategories of \mathcal{C} . The main technique we use here is the biset decomposition (see [6]).

The first special case we consider is the following category \mathcal{D} : $\text{Ob } \mathcal{D} = \text{Ob } \mathcal{C}$, $\mathcal{D}(x, x) = G_1$, $\mathcal{D}(y, y) = H$, and $\mathcal{D}(x, y) = H\alpha G_1 = H\alpha$. We picture the structure of \mathcal{D} below:

$$\mathcal{D} : \begin{array}{ccc} \textcircled{G_1} & \xrightarrow{H\alpha G_1} & \textcircled{H} \\ x & \xrightarrow{\dots} & y \end{array}$$

Lemma 6.3. Let \mathcal{D} be defined as above. Then $k\mathcal{D} \mid ({}_k\mathcal{D}k\mathcal{C}_k\mathcal{D})$. In particular, \mathcal{C} is of infinite representation type if so is \mathcal{D} .

Proof. Let $g_1 = 1, \dots, g_m$ be a chosen set of representatives of the right cosets $G_1 \backslash G$. Clearly,

$$\text{Mor } \mathcal{C} = G \sqcup H \sqcup H\alpha G = \left(\bigsqcup_{i=1}^m G_1 g_i \right) \sqcup H \sqcup \left(\bigcup_{i=1}^m H\alpha g_i \right).$$

We claim that $\bigcup_{i=1}^m H\alpha g_i$ is a disjoint union. If this is not true, we can find some $1 \leq i \neq j \leq m$ such that $H\alpha g_i \cap H\alpha g_j \neq \emptyset$. In particular, there exist $h_1, h_2 \in H$ such that $h_1\alpha g_i = h_2\alpha g_j$. Thus $h_2^{-1}h_1\alpha = \alpha g_j g_i^{-1}$, and $g_j g_i^{-1} \in G_1$. Consequently, g_i and g_j are in the same coset. This is a contradiction.

Define $R_1 = k(G_1 \cup H\alpha)$, $R_2 = k(G_1 g_2 \cup H\alpha g_2)$, \dots , $R_m = k(G_1 g_m \cup H\alpha g_m)$. They are all left $k\mathcal{D}$ -modules. By the above biset decomposition, we get:

$${}_k\mathcal{D}k\mathcal{C} = kH \oplus R_1 \oplus R_2 \oplus \dots \oplus R_m = k\mathcal{D} \oplus R$$

since $R_1 \oplus kH = k\mathcal{D}$.

The above decomposition ${}_k\mathcal{D}k\mathcal{C} = k\mathcal{D} \oplus R$ is also a decomposition of $k\mathcal{C}_k\mathcal{D}$. Indeed, $R = (\bigoplus_{i=2}^m kG_1 g_i) \oplus (\bigoplus_{i=2}^m kH\alpha g_i)$ as a k -vector space. But both $\bigoplus_{i=2}^m kG_1 g_i$

and $\bigoplus_{i=2}^m kH\alpha g_i$ are right $k\mathcal{D}$ -modules. So R is a right $k\mathcal{D}$ -module as well. This proves the first statement. The second one follows from Proposition 3.2. \square

Now consider the second special case. We introduce some notation here. Let $K \leq G_1$ be a subgroup containing G_0 . Then $\alpha K \subseteq \alpha G_1 \subseteq H\alpha$. Define $L = \text{Stab}_H(\alpha K)$. That is, $L = \{h \in H \mid h\alpha K = \alpha K\} = \{h \in H \mid \exists g \in K \text{ such that } h\alpha = \alpha g\}$.

Lemma 6.4. *Notation as above. Then L is a subgroup of H_1 , and $L/H_0 \cong K/G_0$.*

Proof. We can check that L is a subgroup of H_1 as we did in the proof of Lemma 3.3 in [14]. Since $H_0 \triangleleft H_1$ by that lemma, we know $H_0 \triangleleft L$. The fact that $L/H_0 \cong K/G_0$ can be shown as in Lemma 3.3 of [14], too. \square

Thus $H_0 \triangleleft L \leq H_1 \leq H$ and $G_0 \triangleleft K \leq G_1 \leq G$. We then construct a subcategory \mathcal{E} as follows:

$$\mathcal{E} : \quad \begin{array}{ccc} \curvearrowright & \xrightarrow{L\alpha K} & \curvearrowright \\ K & x \xrightarrow{\quad \dots \quad} y & K \end{array}$$

The reader can check that both L and K act transitively on $\mathcal{E}(x, y)$ since $L\alpha K = L\alpha = \alpha K$ by the definitions of K and L .

Lemma 6.5. *Let \mathcal{E} be constructed as above and suppose that both G and H act transitively on $\mathcal{C}(x, y)$. Then $k\mathcal{E} \mid ({}_k\mathcal{E}k\mathcal{C}_k\mathcal{E})$. In particular, \mathcal{C} is of infinite representation type if so is \mathcal{E} .*

Proof. Choose a representative from each coset in $K \backslash G$: $g_1 = 1, \dots, g_m$. Since both G and H acts transitively on $\mathcal{C}(x, y)$, we know that $G = G_1$ and $H = H_1$. Moreover, $G_1/G_0 \cong H_1/H_0$. For each g_i , $1 \leq i \leq m$, there is some h_i such that $\alpha g_i = h_i \alpha$. We claim that $Lh_i \alpha \cap Lh_j \alpha = \emptyset$, and hence $Lh_i \cap Lh_j = \emptyset$, $1 \leq i \neq j \leq m$. Indeed, since $Lh_i \alpha = L\alpha g_i = \alpha K g_i$ and $Lh_j \alpha = L\alpha g_j = \alpha K g_j$. If the intersection is not empty, there exist $u, v \in K$ such that $\alpha u g_i = \alpha v g_j$, i.e., $\alpha = \alpha v g_j g_i^{-1} u^{-1}$. Therefore, $v g_j g_i^{-1} u^{-1} \in G_0 \leq K$, and hence $g_j g_i^{-1} \in K$. This is impossible since g_i and g_j are in different cosets.

Note that $\{h_i\}_{i=1}^m$ is a set of representatives of the cosets $L \backslash H$ since $G/G_0 \cong H/H_0$ and $K/G_0 \cong L/H_0$. Clearly,

$$\text{Mor } \mathcal{C} = G \sqcup H \sqcup H\alpha = \left(\bigsqcup_{i=1}^m K g_i \right) \sqcup \left(\bigsqcup_{j=1}^m L h_j \right) \sqcup \left(\bigsqcup_{j=1}^m L h_j \alpha \right).$$

The vector space spanned by $K \sqcup L\alpha \sqcup L$ is precisely $k\mathcal{E}$. Let R be the vector space spanned by $(\bigsqcup_{i=2}^m K g_i) \sqcup (\bigsqcup_{j=2}^m L h_j) \sqcup (\bigsqcup_{j=2}^m L h_j \alpha)$. Then R is a left $k\mathcal{E}$ -module, and hence ${}_k\mathcal{E}k\mathcal{C} = k\mathcal{E} \oplus R$. We also have $k\mathcal{C}_k\mathcal{E} = k\mathcal{E} \oplus R$. Indeed, since

$$\bigsqcup_{j=2}^m L h_j \alpha K = \bigsqcup_{j=2}^m L h_j L \alpha = \bigsqcup_{j=2}^m L h_j \alpha,$$

we check that R is a right $k\mathcal{E}$ -module as well. This proves the first statement. The second one follows from Proposition 3.2. \square

Given a subcategory \mathcal{D} of \mathcal{C} as pictured below and $N \in k\mathcal{D}\text{-mod}$, where $G' = \mathcal{D}(x, x)$ and $H' = \mathcal{D}(y, y)$.

$$\mathcal{D} : \quad \begin{array}{ccc} \curvearrowright & \xrightarrow{H' \alpha G'} & \curvearrowright \\ G' & x \xrightarrow{\quad \dots \quad} y & H' \end{array}$$

We want to describe the structure of the induced module $N \uparrow_{\mathcal{D}}^{\mathcal{C}}$. As described in Proposition 6.1, N is determined by a linear map $\varphi : V \rightarrow W$ where $V = N(x)$

and $W = N(y)$ are a kG' -module and a kH' -module respectively. In general, $N \uparrow_{\mathcal{D}}^{\mathcal{C}}(x) \not\cong V \uparrow_{G'}^{\mathcal{C}}$, and $N \uparrow_{\mathcal{D}}^{\mathcal{C}}(y) \not\cong W \uparrow_{H'}^{\mathcal{C}}$. Indeed, we have already seen in Example 3.1 that the dimension of the induced module can be even smaller than that of the original module. In the following example we see that the induced module is isomorphic to the original one.

Example 6.6. Let \mathcal{C} be a finite EI category such that: $\text{Ob}\mathcal{C} = \{x, y\}$, $G = 1$, $H = \langle h \rangle$ is a cyclic group of order 2 = $\text{char}(k)$, and $\mathcal{C}(x, y) = \{\alpha\}$. Let \mathcal{D} be the subcategory formed by removing $h \in H$. Let $\varphi : V \rightarrow W$ be a representation R of \mathcal{D} , where $V = \langle v \rangle \cong k$, $W = \langle w \rangle \cong k$ and $\varphi(v) = w$. By direct computation we get $R \uparrow_{\mathcal{D}}^{\mathcal{C}}(x) \cong V$ and $R \uparrow_{\mathcal{D}}^{\mathcal{C}}(y) \cong W \not\cong W \uparrow_1^H$:

$$h \otimes_{k\mathcal{D}} w = h \otimes_{k\mathcal{D}} \alpha \cdot v = h\alpha \otimes_{k\mathcal{D}} v = \alpha \otimes_{k\mathcal{D}} v = 1_y \otimes_{k\mathcal{D}} \alpha \cdot v = 1_y \otimes w.$$

If the pair $(\mathcal{C}, \mathcal{D})$ satisfies some extra property, we can explicitly describe the structure of the induced module $R \uparrow_{\mathcal{D}}^{\mathcal{C}}$ by a linear map between two induced group representations.

Lemma 6.7. Suppose that H acts transitively on $\mathcal{C}(x, y)$. Let \mathcal{D} be a subcategory of \mathcal{C} pictured as above and R be a $k\mathcal{D}$ -module of the form $R(\alpha) = \varphi : V \rightarrow W$. If $\text{Stab}_H(\alpha G') \leq H'$, then $\tilde{R} = R \uparrow_{\mathcal{D}}^{\mathcal{C}}$ has the form

$$\tilde{R}(\alpha) = \tilde{\varphi} : kG \otimes_{kG'} V \rightarrow kH \otimes_{kH'} W, \quad g \otimes v \mapsto h \otimes \varphi(v),$$

where $h \in H$ such that $\alpha g = h\alpha$.

Proof. We first check that $\tilde{\varphi}$ indeed determines a $k\mathcal{C}$ -module. Since $kG \otimes_{kG'} V$ and $kH \otimes_{kH'} W$ have a natural kG -module structure and a natural kH -module structure respectively, it is enough to prove that $h_1 \tilde{\varphi} g_1$ and $h_2 \tilde{\varphi} g_2$ coincide as linear maps from $kG \otimes_{kG'} V$ to $kH \otimes_{kH'} W$ if $h_1 \alpha g_1 = h_2 \alpha g_2$, $g_1, g_2 \in G$, $h_1, h_2 \in H$.

Take $g \otimes_{kG'} v \in kG \otimes_{kG'} V$. Since H acts transitively, we can find some $h \in H$ such that $h\alpha = \alpha g$. Moreover, $h_1 \alpha g_1 = h_2 \alpha g_2$ implies $\alpha g_1 = h_1^{-1} h_2 \alpha g_2$, so $\alpha g_1 g = h_1^{-1} h_2 \alpha g_2 g$. Therefore,

$$h_1 \tilde{\varphi} g_1(g \otimes_{kG'} v) = h_1 \tilde{\varphi}(g_1 g \otimes_{kG'} v) = h_1 h \otimes_{kH'} \varphi(v),$$

where $h\alpha = \alpha g_1 g = h_1^{-1} h_2 \alpha g_2 g$. On the other hand, we also have

$$h_2 \tilde{\varphi} g_2(g \otimes_{kG'} v) = h_2 \tilde{\varphi}(g_2 g \otimes_{kG'} v) = h_2 h' \otimes_{kH'} \varphi(v),$$

where $h'\alpha = \alpha g_2 g$. Thus $h\alpha = h_1^{-1} h_2 \alpha g_2 g = h_1^{-1} h_2 h' \alpha$, and $\alpha = h^{-1} h_1^{-1} h_2 h' \alpha$. Consequently, $h^{-1} h_1^{-1} h_2 h' \in H_0 \triangleleft H' \leq H_1$. By Proposition 6.1, H_0 acts trivially on $\varphi(v)$, so we have

$$h_1 h \otimes_{kH'} \varphi(v) = h_1 h \otimes_{kH'} h^{-1} h_1^{-1} h_2 h' \varphi(v) = h_2 h' \otimes_{kH'} \varphi(v)$$

as required. Therefore, $\tilde{\varphi}$ is a well defined representation of \mathcal{C} .

Now Let $\{v_i\}_{i=1}^m$ and $\{w_j\}_{j=1}^n$ be bases of V and W respectively. Note that every morphism in $\mathcal{C}(x, y)$ can be written as $h\alpha$ for some $h \in H$. We define a k -linear map $\pi : k\mathcal{C} \times R \rightarrow \tilde{R}$ by:

$$\begin{aligned} (g, v_i) &\mapsto g \otimes_{kG'} v_i, & (h\alpha, v_i) &\mapsto h \otimes_{kH'} \varphi(v_i), & (h, w_j) &\mapsto h \otimes_{kH'} w_j \\ (h, v_i) &\mapsto 0, & (\alpha, w_j) &\mapsto 0, & (g, w_j) &\mapsto 0 \end{aligned}$$

for $g \in G, h \in H, 1 \leq i \leq m, 1 \leq j \leq n$. The reader can check that π is $k\mathcal{D}$ -balanced using either the condition $\text{Stab}_H(\alpha G') \leq H'$, so it induces the following $k\mathcal{D}$ -homomorphism $\tilde{\pi} : R \uparrow_{\mathcal{D}}^{\mathcal{C}} \rightarrow \tilde{R}$:

$$g \otimes_{k\mathcal{D}} v_i \mapsto g \otimes_{kG'} v_i, \quad h \otimes_{k\mathcal{D}} w_j \mapsto h \otimes_{kH'} w_j$$

for $g \in G, h \in H, 1 \leq i \leq m, 1 \leq j \leq n$, as shown by the following commutative diagram:

$$\begin{array}{ccc} g \otimes_{k\mathcal{D}} v_i & \xrightarrow{\quad} & \alpha g \otimes_{k\mathcal{D}} v_i = h\alpha \otimes_{k\mathcal{D}} v_i = h \otimes_{k\mathcal{D}} \varphi(v_i) \\ \downarrow \tilde{\pi}_x & & \downarrow \tilde{\pi}_y \\ g \otimes_{kG'} v_i & \xrightarrow{\tilde{\varphi}} & \tilde{\varphi}(g \otimes_{kG'} v_i) = h \otimes_{kH'} \varphi(v_i) \end{array}$$

Note that $\tilde{\pi}$ is not only a $k\mathcal{D}$ -module homomorphism, but also a $k\mathcal{C}$ -module homomorphism. Furthermore, from our definition of π we find that all basis elements of \tilde{R} are images under π . By the universal property of tensor product, they are actually images under $\tilde{\pi}$. Therefore, $\tilde{\pi}$ is a surjective $k\mathcal{C}$ -module homomorphism.

On the other hand, $R \uparrow_{\mathcal{D}}^{\mathcal{C}}(x)$ is spanned by basic tensors $g \otimes_{k\mathcal{D}} v_i, 1 \leq i \leq m$. Since $kG' \subset k\mathcal{D}$, it is actually spanned by the set $\{g_t \otimes_{k\mathcal{D}} v_i \mid 1 \leq t \leq l, 1 \leq i \leq m\}$, where $\{g_t\}_{t=1}^l$ is a set of representatives of the left cosets G/G' . It turns out that

$$\dim_k(R \uparrow_{\mathcal{D}}^{\mathcal{C}}(x)) \leq |G : G'| \dim_k V = \dim_k \tilde{R}(x).$$

Similarly, since H acts transitively on $\mathcal{C}(x, y)$, any morphism β in $\mathcal{C}(x, y)$ can be written as $h\alpha$ for some $h \in H$. Thus $\beta(g \otimes_{k\mathcal{D}} v_i) = h\alpha g \otimes_{k\mathcal{D}} v_i = hh' \otimes_{k\mathcal{D}} \varphi(v_i)$ where $\alpha g = h'\alpha$. Therefore, $R \uparrow_{\mathcal{D}}^{\mathcal{C}}(y)$ is spanned by basic tensors $h \otimes_{k\mathcal{D}} w_j, 1 \leq j \leq n, h \in H$. Since $kH' \subset k\mathcal{D}$, it is actually spanned by the set $\{h_s \otimes_{k\mathcal{D}} w_j \mid 1 \leq s \leq r, 1 \leq j \leq n\}$, where $\{h_s\}_{s=1}^r$ is a set of representatives of the cosets H/H' . Therefore,

$$\dim_k(R \uparrow_{\mathcal{D}}^{\mathcal{C}}(y)) \leq |H : H'| \dim_k W = \dim_k \tilde{R}(y).$$

By the above two inequalities, we have $\dim_k(R \uparrow_{\mathcal{D}}^{\mathcal{C}}) \leq \dim_k \tilde{R}$, so $\tilde{\pi}$ is actually an isomorphism. \square

We will use the following result to get a family of finite EI categories of finite representation type.

Proposition 6.8. *Let \mathcal{D} be as in the previous lemma. Suppose that both $|G : G'|$ and $|H : H'|$ are invertible in k . If the pair $(\mathcal{C}, \mathcal{D})$ satisfies one of the following conditions:*

- (1) G acts transitively on $\mathcal{C}(x, y)$;
- (2) $H' = H$,

then M is isomorphic to a direct summand of $M \downarrow_{\mathcal{D}}^{\mathcal{C}} \uparrow_{\mathcal{D}}^{\mathcal{C}}$ for $M \in k\mathcal{C}\text{-mod}$. In particular, \mathcal{C} has finite representation type if so does \mathcal{D} .

Proof. The second statement follows from Proposition 3.2, so we only need to show the first statement. Firstly we consider a special case. That is, G acts transitively on $\mathcal{C}(x, y)$ and $G' = \text{Stab}_G(H'\alpha)$. Observe that by this given condition $G_0 \triangleleft G_1 = G$, $H_1 \triangleleft H_1 = H$, and $H/H_0 \cong G/G_0$. Moreover, by Lemma 6.4, $G'/G_0 \cong H'/H_0$. Take $\{g_i\}_{i=1}^n$ as a set of representatives of the cosets G/G' and choose $h_i \in H$ satisfying $\alpha g_i = h_i \alpha$. Then $\{h_i\}_{i=1}^n$ is a set of representatives of the cosets H/H' , and $\alpha g_i G' = h_i H' \alpha$. Note that n is invertible.

Suppose that M has the form $\varphi : V \rightarrow W$ where: $V = M(x)$ is a kG -module, $W = M(y)$ is a kH -module, and $\varphi = M(\alpha)$. By the previous lemma, $M \downarrow_{\mathcal{D}}^{\mathcal{C}} \uparrow_{\mathcal{D}}^{\mathcal{C}}$ has the form

$$\tilde{\varphi} : kG \otimes_{kG'} V \rightarrow kH \otimes_{kH'} W, \quad g \otimes v \mapsto h \otimes \varphi(v),$$

where $h\alpha = \alpha g$.

Define the following maps:

$$\begin{aligned} \theta_V : kG \otimes_{kG'} V &\rightarrow V, & g \otimes v &\mapsto gv; \\ \delta_V : V &\rightarrow kG \otimes_{kG'} V, & v &\mapsto \frac{1}{n} \sum_{i=1}^n g_i \otimes g_i^{-1} v; \\ \theta_W : kH \otimes_{kH'} W &\rightarrow W, & h \otimes w &\mapsto hw; \\ \delta_W : W &\rightarrow kH \otimes_{kH'} W, & w &\mapsto \frac{1}{n} \sum_{i=1}^n h_i \otimes h_i^{-1} w. \end{aligned}$$

and consider the diagram

$$\begin{array}{ccc} kG \otimes_{kG'} V & \xrightarrow{\tilde{\varphi}} & kH \otimes_{kH'} W \\ \delta_V \updownarrow \theta_V & & \delta_W \updownarrow \theta_W \\ V & \xrightarrow{\varphi} & W. \end{array}$$

We can show that the diagram commutes. Indeed, for $g \in G$ and $v \in V$,

$$\begin{aligned} \varphi \theta_V(g \otimes v) &= \varphi(gv) = (\alpha g) \cdot v = (h\alpha) \cdot v \\ &= h\varphi(v) = \theta_W(h \otimes \varphi(v)) = \theta_W \tilde{\varphi}(g \otimes v) \end{aligned}$$

where $\alpha g = h\alpha$; and

$$\begin{aligned} \tilde{\varphi} \delta_V(v) &= \frac{1}{n} \sum_{i=1}^n \tilde{\varphi}(g_i \otimes g_i^{-1} v) = \frac{1}{n} \sum_{i=1}^n h_i \otimes \varphi(g_i^{-1} v) = \frac{1}{n} \sum_{i=1}^n h_i \otimes (\alpha g_i^{-1}) \cdot v \\ &= \frac{1}{n} \sum_{i=1}^n h_i \otimes h_i^{-1} \alpha \cdot v = \frac{1}{n} \sum_{i=1}^n h_i \otimes h_i^{-1} \varphi(v) = \delta_W \varphi(v). \end{aligned}$$

But $\theta_V \delta_V = 1_V$ and $\theta_W \delta_W = 1_W$. Therefore, $M \mid M \downarrow_{\mathcal{D}}^{\mathcal{C}} \uparrow_{\mathcal{D}}^{\mathcal{C}}$.

Now suppose that \mathcal{D} satisfies the second condition, i.e., $H' = H$. Thus $\mathcal{D}(x, y) = H\alpha = \mathcal{C}(x, y)$. By the previous lemma, $M \downarrow_{\mathcal{D}}^{\mathcal{C}} \uparrow_{\mathcal{D}}^{\mathcal{C}}$ has the form

$$\tilde{\varphi} : kG \otimes_{kG'} V \rightarrow kH \otimes_{kH} W \cong W, \quad g \otimes v \mapsto h \otimes \varphi(v) = 1 \otimes h\varphi(v),$$

where $\alpha g = h\alpha$.

Let $\{g_i\}_{i=1}^n$ be a set of representatives of the cosets G/G' . Define θ_V and δ_V as in the first case. We can show the following diagram commutes:

$$\begin{array}{ccc} kG \otimes_{kG'} V & \xrightarrow{\tilde{\varphi}} & kH \otimes_{kH} W \\ \delta_V \updownarrow \theta_V & & \cong \updownarrow \cong \\ V & \xrightarrow{\varphi} & W. \end{array}$$

So the conclusion follows as in the above special case.

In the situation that \mathcal{D} satisfies the first condition, i.e., G acts transitively on $\mathcal{C}(x, y)$, we define another subcategory \mathcal{E} of \mathcal{C} as follows:, where $\hat{G} = \text{Stab}_G(H'\alpha)$:

$$\mathcal{E} : \quad \begin{array}{ccc} \curvearrowright_G x & \xrightarrow[H' \alpha \hat{G}]{\dots} & y \curvearrowright_{H'} \end{array}$$

Note that both \hat{G} and H' act transitively on $\mathcal{E}(x, y)$, so $\text{Stab}_H(\alpha \hat{G}) = H'$.

The pair $(\mathcal{C}, \mathcal{E})$ falls into the special case we consider at the beginning, so we have $M \mid M \downarrow_{\mathcal{E}}^{\mathcal{C}} \uparrow_{\mathcal{E}}^{\mathcal{C}}$. The pair $(\mathcal{E}, \mathcal{D})$ satisfies the second condition, so we have $(M \downarrow_{\mathcal{E}}^{\mathcal{C}}) \mid (M \downarrow_{\mathcal{D}}^{\mathcal{E}}) \downarrow_{\mathcal{D}}^{\mathcal{E}} \uparrow_{\mathcal{D}}^{\mathcal{E}}$. In conclusion, we get $M \mid M \downarrow_{\mathcal{E}}^{\mathcal{C}} \downarrow_{\mathcal{D}}^{\mathcal{E}} \uparrow_{\mathcal{D}}^{\mathcal{E}} \uparrow_{\mathcal{E}}^{\mathcal{C}}$, i.e., $M \mid M \downarrow_{\mathcal{D}}^{\mathcal{C}} \uparrow_{\mathcal{D}}^{\mathcal{C}}$. This finishes the proof. \square

6.3. Transitivity of group actions. Our main task in this subsection is to prove the following proposition, which is condition (2) in Theorem 1.1.

Proposition 6.9. *If neither G nor H acts transitively on $\mathcal{C}(x, y)$, then \mathcal{C} is of infinite representation type.*

This conclusion has been proved in [14] for the case that both $|G|$ and $|H|$ are invertible. We prove the fact for the general case. Define a finite EI category \mathcal{Q} as follows, where $\text{Stab}_G(\bar{\alpha}) = G_1$ and $\text{Stab}_H(\bar{\alpha}) = H_1$.

$$\mathcal{Q} : \quad \begin{array}{ccc} \curvearrowright_G x & \xrightarrow[H \bar{\alpha} G]{\dots} & y \curvearrowright_H \end{array}$$

We claim that \mathcal{Q} is a quotient category of \mathcal{C} . Indeed, let $F : \mathcal{C} \rightarrow \mathcal{Q}$ be the functor such that F is the identity map on objects and automorphisms, and sends $h\alpha g$ to $h\bar{\alpha}g$ for $h \in H$ and $g \in G$. If $h_1\alpha g_1 = h_2\alpha g_2$, $h_1, h_2 \in H$, $g_1, g_2 \in G$, then $h_2^{-1}h_1\alpha = \alpha g_2 g_1^{-1}$, so $h_2^{-1}h_1 \in H_1$ and $g_2 g_1^{-1} \in G_1$. Therefore, $h_2^{-1}h_1\bar{\alpha} = \bar{\alpha} g_2 g_1^{-1} = \bar{\alpha}$. In other words, $h_1\bar{\alpha}g_1 = h_2\bar{\alpha}g_2$. Thus F is a well defined quotient functor, and \mathcal{Q} is a quotient category of \mathcal{C} . Therefore, it suffices to show the infinite representation type of \mathcal{Q} . The strategy is to use permutation modules $M = k \uparrow_{G_1}^G$ and $N = k \uparrow_{H_1}^H$ to construct infinitely many distinct indecomposable representations (up to isomorphism) for \mathcal{Q} . Since neither G nor H acts transitively on $\mathcal{C}(x, y)$, we know that $G_1 \neq G$ and $H_1 \neq H$, so $\dim_k M > 1$ and $\dim_k N > 1$.

Note that M has a unique indecomposable summand M_0 which is characterized by one of the following properties:

- (1) $\text{Hom}_{kG}(M_0, k) \cong k$, i.e, $\text{Top}(M_0)$ has a unique summand isomorphic to k ;
- (2) $\text{Hom}_{kG}(k, M_0) \cong k$, i.e, $\text{Soc}(M_0)$ has a unique summand isomorphic to k .

Moreover, M_0 is isomorphic to its dual, and $\text{Top}(M_0) \cong \text{Soc}(M_0)$. This module is called a *Scott module*. In one extreme situation that H_1 contains a Sylow p -subgroup of H , $M_0 \cong k$. If $|H_1|$ is invertible, then $M_0 \cong P_k$, the projective cover of k . For more details, see [7, 13]. In particular, for a finite group with cyclic Sylow p -subgroups (as we study in this paper), the structure of all Scott modules has been described in [20] by considering the Brauer graph of the principal block. Similarly, $N = k \uparrow_{H_1}^H$ has a Scott module N_0 .

Proof. We prove the conclusion by considering the permutation modules M and N . There are three cases:

Case I: Both M and N have at least two indecomposable summands. Take an indecomposable summand M_1 of M which is not the Scott module, and let S be a simple summand of $\text{Soc}(M_1)$. Clearly, $S \not\cong k$. Moreover, since $\text{Hom}_{kG}(S, M_1) \neq 0$,

we deduce that $\text{Hom}_{kG_1}(S \downarrow_{G_1}^G, k) \cong \text{Hom}_{kG}(S, M) \neq 0$, so $\text{Top}(S \downarrow_{G_1}^G)$ has a simple summand $k_1 \cong k$. Dually, we can find a simple kH -module $T \not\cong k$ such that $\text{Soc}(T \downarrow_{H_1}^H)$ has a simple summand $k_2 \cong k$.

As vector spaces, $S = k_1 \oplus S'$ and $T = k_2 \oplus T'$ (S' and T' might be 0, but this is fine). Construct a family of representations R_d of \mathcal{Q} as follows, $0 \neq d \in k$. First, $R_d(x) = k \oplus S = k \oplus (k_1 \oplus S')$, $R_d(y) = k \oplus T = k \oplus (k_2 \oplus T')$. The linear map $R_d(\bar{\alpha})$ is defined by:

$$k \oplus k_1 \oplus S' \xrightarrow{\begin{bmatrix} 1 & 1 & 0 \\ 1 & d & 0 \\ 0 & 0 & 0 \end{bmatrix}} k \oplus k_2 \oplus T'$$

We check that this linear map $R_d(\bar{\alpha})$ indeed gives rise to a representation of \mathcal{Q} by Proposition 6.1, and that R_d is indecomposable by computing its endomorphism algebra. Moreover, if $R_b \cong R_d$, we have the following matrix identity:

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & d & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & v \end{bmatrix} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & t \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & b & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where $b, d, u, t, v, \lambda \in k$ are nonzero scalars. By computation, we get $b = d$. In this way we construct infinitely many pairwise non-isomorphic indecomposable representations of \mathcal{Q} illustrated as follows, where dotted arrows means that these maps are actually only defined on some subspaces.

$$\begin{array}{ccc} k & \xrightarrow{1} & k \\ & \searrow & \nearrow \\ & 1 & \\ & \swarrow & \searrow \\ S & \xrightarrow{d} & T. \end{array}$$

Case II: One of M and N is indecomposable and the other one is not. Without loss of generality we assume that M is indecomposable. Then $M \not\cong k$ is the Scott module. Its length is at least 2; and both $\text{Top}(M)$ and $\text{Soc}(M)$ has a unique composition factor isomorphic to k . Let k_1 be this composition factor in $\text{Top}(M)$.

Observe that $\dim_k \text{End}_{kG}(M) \geq 2$ since there is a nilpotent map $\varphi : M \rightarrow M$ such that $\varphi(M) \cong k$ is a simple summand of $\text{Soc}(M)$. We conclude that

$$\dim_k \text{Hom}_{kG_1}(M \downarrow_{G_1}^G, k) = \dim_k \text{Hom}_{kG}(M, k \uparrow_{G_1}^G) = \dim_k \text{Hom}_{kG}(M, M) \geq 2.$$

This fact also follows from

$$\begin{aligned} \dim_k \text{Hom}_{kG_1}(M \downarrow_{G_1}^G, k) &= \dim_k \text{Hom}_{kG_1}(k \uparrow_{G_1}^G \downarrow_{G_1}^G, k) \\ &= \dim_k \text{Hom}_{kG_1} \left(\bigoplus_{s \in G_1 \backslash G / G_1} (s \otimes k) \uparrow_{G_1 \cap sG_1s^{-1}}^{G_1}, k \right). \end{aligned}$$

The last dimension equals the number of double cosets $G_1 \backslash G / G_1$, which is at least 2. Therefore, there are at least two summands isomorphic to k in $\text{Top}(M \downarrow_{G_1}^G)$. Take a summand $k_2 \cong k$ in $\text{Top}(M \downarrow_{G_1}^G)$ which is different from k_1 .

Write $M = k_1 \oplus k_2 \oplus M'$ as vector spaces. Take a simple kH -module T as we did in Case I. Note that $T \downarrow_{H_1}^H = k_3 \oplus T'$ where $k_3 \cong k$. We construct a class of representations R_d in the following way, $0 \neq d \in k$. First, $R_d(x) = M$,

$R_d(y) = k \oplus T$. The map $R_d(\bar{\alpha})$ is defined by

$$k_1 \oplus k_2 \oplus M' \xrightarrow{\begin{bmatrix} 1 & 1 & 0 \\ 1 & d & 0 \\ 0 & 0 & 0 \end{bmatrix}} k \oplus k_3 \oplus T'.$$

Again, by Proposition 6.1, $R_d(\bar{\alpha})$ gives rise to a representation of \mathcal{Q} . It is indecomposable since $R(x) = M$ is an indecomposable kG -module. Moreover, $R_d \cong R_b$ if and only if $d = b$. Indeed, if $R_b \cong R_d$, we have the following matrix identity:

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & d & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u & 0 & 0 \\ \lambda & u & \varphi_{32} \\ \varphi_{13} & \varphi_{23} & \varphi_{33} \end{bmatrix} = \begin{bmatrix} v & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & t \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & b & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

In the above identity, $b, d, u, t, v \in k$ are nonzero scalars, $\lambda \in k$, and $\varphi_{33} = uI + \delta$, where I is the identity matrix and δ is a nilpotent matrix. By computation, we get $b = d$. In this way we construct infinitely many pairwise non-isomorphic indecomposable representations of \mathcal{Q} .

Case III: Both M and N are indecomposable. As in Case II, there exist at least two summands $k_1 \cong k_2 \cong k$ in $\text{Top}(M \downarrow_{G_1}^G)$, where k_1 is the restriction of the simple summand k in $\text{Top}(M)$ to G_1 . Dually, there exist at least two summands $k_3 \cong k_4 \cong k$ in $\text{Soc}(N \downarrow_{H_1}^H)$, where k_3 is the restriction of the simple summand k in $\text{Soc}(N)$ to H_1 .

Write $M = k_1 \oplus k_2 \oplus M_1$ and $N = k_3 \oplus k_4 \oplus N_1$ as vector spaces. Then the following map φ

$$k_1 \oplus k_2 \oplus M_1 \xrightarrow{\begin{bmatrix} 1 & 1 & 0 \\ 1 & d & 0 \\ 0 & 0 & 0 \end{bmatrix}} k_3 \oplus k_4 \oplus N_1$$

gives rise to a representation R_d of \mathcal{Q} by Proposition 6.1. It is obviously indecomposable. Moreover, letting $d \neq 0$ vary in k , we check similarly that $R_d \cong R_b$ if and only if $d = b$. \square

This proposition implies Theorem 3.35 in [10], which asserts that if both H and G are nontrivial and $\mathcal{C}(x, y) \cong H \times G$ as a (H, G) -biset, then \mathcal{C} is of infinite representation type.

6.4. Actions of p -subgroups. In this subsection we set $\text{char}(k) = p \geq 5$. It is well known that the representation type of a finite group is completely determined by its Sylow p -subgroups. For the finite EI category \mathcal{C} , the Sylow p -subgroups of G and H still play an important role in determining the representation type of \mathcal{C} as illustrated by conditions (3) and (4) in Theorem 1.1, which will be proved in this subsection.

By the previous proposition, throughout this subsection without loss of generality we assume that H **acts transitively** on $\mathcal{C}(x, y)$. Observe that the condition that H has a p -subgroup acting nontrivially on $\mathcal{C}(x, y)$ is equivalent to one of the following conditions:

- (1) H has a Sylow p -subgroup acting nontrivially on $\mathcal{C}(x, y)$;
- (2) all Sylow p -subgroups of H act nontrivially on $\mathcal{C}(x, y)$;

- (3) $O^{p'}H$, the normal subgroup generated by all Sylow p -subgroups of H , is not contained in H_0 ;
- (4) there is some $h \in H$ with order a power of p such that $h\alpha \neq \alpha$.

Moreover, if G has a p -subgroup acting nontrivially on $\mathcal{C}(x, y)$, so does H . Indeed, since H acts transitively, $G_0 \triangleleft G_1 = G$. Consequently, G_0 does not contain a Sylow p -subgroup of G since otherwise $O^{p'}G \leq G_0$, contradicting the given condition. Thus p divides $|G_1/G_0| = |H_1/H_0|$, so H_0 does not contain a Sylow p -subgroup of $H_1 \leq H$. In conclusion, H_1 and H have a p -subgroup acting nontrivially on $\mathcal{C}(x, y)$.

Proposition 6.10. *Suppose that $\text{char}(k) = p \geq 5$. If both G and H have p -subgroups acting nontrivially on $\mathcal{C}(x, y)$, then \mathcal{C} is of infinite representation type.*

Proof. We have assumed that H acts transitively on $\mathcal{C}(x, y)$. Consequently, $G = G_1$. Define a subcategory \mathcal{D} of \mathcal{C} as follows:

$$\mathcal{D} : \quad \begin{array}{ccc} \curvearrowright_G x & \xrightleftharpoons[H_1\alpha G]{\dots} & y \curvearrowright_{H_1} \end{array}$$

By Lemma 6.3, it suffices to show the infinite representation type of \mathcal{D} .

We claim that both G and H_1 act transitively on $\mathcal{D}(x, y) = H_1\alpha G$. Indeed, by the definition of H_1 , $H_1\alpha \subseteq \alpha G$, so $H_1\alpha G = \alpha G$, i.e., G acts transitively on $\mathcal{D}(x, y)$. On the other hand, since H acts transitively on $\mathcal{C}(x, y)$, for every $g \in G$, there exists some $h \in H$ with $h\alpha = \alpha g$. But again by the definition of H_1 , $h \in H_1$. Therefore, $\alpha G \subseteq H_1\alpha$. This proves the claim.

Let $\bar{G} = G/G_0$ and $\bar{H} = H_1/H_0$. Since $\bar{G} \cong \bar{H}$, we identify these two groups. Moreover, p divides $|\bar{G}| = |\bar{H}|$. Indeed, since G has a p -subgroup acting nontrivially on $\mathcal{C}(x, y)$, we can find some $g \in G$ with order a power of p such that $\alpha g \neq \alpha$, i.e., $g \notin G_0$. Therefore, its image $\bar{g} \in \bar{G}$ has order a power of p , so p divides $|\bar{G}|$.

Define another finite EI category \mathcal{E} as follows:

$$\mathcal{E} : \quad \begin{array}{ccc} \curvearrowright_{\bar{G}} x & \xrightleftharpoons[\bar{H}\bar{\alpha}\bar{G}]{\dots} & y \curvearrowright_{\bar{H}} \end{array}$$

where \bar{G} acts regularly on $\mathcal{E}(x, y)$ from the right side and \bar{H} acts regularly on it from the left side. The reader can check that \mathcal{E} is a quotient category of \mathcal{D} by factoring out G_0 and H_0 . Consequently, it is enough to show the infinite representation type of \mathcal{E} .

Since p divides $|\bar{G}|$, we can take a nontrivial p -subgroup $P \leq \bar{G}$. Correspondingly, $Q = \text{Stab}_{\bar{H}}(\bar{\alpha}P) \leq \bar{H}$, and $P \cong Q$. Let \mathcal{F} be the following subcategory of \mathcal{E} :

$$\mathcal{F} : \quad \begin{array}{ccc} \curvearrowright_P x & \xrightleftharpoons[Q\bar{\alpha}P]{\dots} & y \curvearrowright_Q \end{array}$$

By Proposition 5.1 \mathcal{F} is of infinite representation type. By Lemma 6.5, \mathcal{E} is of infinite representation type as well. This finishes the proof. \square

By the remark before this proposition, if G has a p -subgroup acting nontrivially on $\mathcal{C}(x, y)$, so does H since it acts transitively. Consequently, \mathcal{C} has infinite representation type. Thus in the rest of this subsection we assume that $O^{p'}G$ **acts trivially on** $\mathcal{C}(x, y)$, i.e., $O^{p'}G \leq G_0$. By this assumption, we know that p does not divide $|G_1 : G_0| = |H_1 : H_0|$. Therefore, for every Sylow p -subgroup $P \leq H$, $P \cap H_0 = P \cap H_1$. In particular, p divides $|H_1|$ if and only if it divides $|H_0|$, and $O^{p'}H \leq H_0$ if and only if $O^{p'}H \leq H_1$.

Lemma 6.11. *Suppose that $G = 1$ and p divides $|H : H_0|$. If $\dim_k \text{End}_{kH}(P_k) \geq 6$ where P_k is the projective cover of the simple kH -module k , then \mathcal{C} is of infinite representation type.*

The condition that p divides $|H : H_0|$ implies that there is a Sylow p -subgroup $P \leq H$ acting nontrivially on $\mathcal{C}(x, y)$. But the converse statement is not true. Indeed, p does not divide $|H : H_0|$ if and only if H_0 contains a Sylow p -subgroup of H , weaker than the condition $O^{p'} \leq H_0$, which by the above remark is equivalent to saying that a Sylow p -subgroup of H acting trivially on $\mathcal{C}(x, y)$. We also remind the reader that P_k can also be viewed as a projective $k\mathcal{C}$ -module and $\text{End}_{k\mathcal{C}}(P_k) \cong \text{End}_{kH}(P_k)$.

Proof. Consider the projective $k\mathcal{C}$ -module $P_k \oplus k\mathcal{C}1_x$ and $\Lambda = \text{End}_{k\mathcal{C}}(P_k \oplus k\mathcal{C}1_x)^{\text{op}}$. By Proposition 2.5 on page 36 in [2], $\Lambda\text{-mod}$ is equivalent to a subcategory of $k\mathcal{C}\text{-mod}$. Thus it suffices to show the infinite representation type of Λ . We have

$$\text{End}_{k\mathcal{C}}(P_k) \cong k[t]/(t^d), \quad \text{End}_{k\mathcal{C}}(k\mathcal{C}1_x) \cong 1_x k\mathcal{C}1_x \cong k, \quad \text{Hom}_{k\mathcal{C}}(k\mathcal{C}1_x, P_k) = 0,$$

where $d = \dim_k \text{End}_{kH}(P_k)$, and

$$\dim_k \text{Hom}_{k\mathcal{C}}(P_k, k\mathcal{C}1_x) = \dim_k \text{Hom}_{kH}(P_k, kH\alpha) = \dim_k \text{Hom}_{kH}(P_k, k \uparrow_{H_0}^H).$$

Let M be the Scott module of $k \uparrow_{H_0}^H$. Since p divides $|H : H_0|$, $M \not\cong k$. Therefore, M has at least two composition factors isomorphic to k . Consequently,

$$t = \dim_k \text{Hom}_{k\mathcal{C}}(P_k, k\mathcal{C}1_x) \geq \dim_k \text{Hom}_{kH}(P_k, M) \geq 2.$$

Thus Λ is isomorphic to the path algebra of the following quiver with relations $\delta^d = 0$ and $\delta^t \beta = 0$ for some $t \geq 2$:

$$\bullet \xrightarrow{\beta} \bullet \begin{array}{c} \curvearrowright \delta \end{array}.$$

From Bongartz's list in [3, 5] we conclude that Λ and hence $k\mathcal{C}$ are of infinite representation type if $d \geq 6$. \square

Before stating and proving the next two propositions, let us do some reduction. Define a finite EI category \mathcal{G} as follows:

$$\mathcal{G} : \quad \begin{array}{c} \textcircled{x} \xrightarrow[\dots]{H\bar{\alpha}} \textcircled{y} \end{array} \begin{array}{c} \curvearrowright H \end{array}.$$

where $\text{Stab}_H(\bar{\alpha}) = H_1$. Let $F : \mathcal{C} \rightarrow \mathcal{G}$ be the functor defined by: $F(g) = 1$, $F(h) = h$, $F(h\alpha g) = h\bar{\alpha}$ for $g \in G$ and $h \in H$. The reader can check that F is well defined. Therefore, \mathcal{G} is a quotient category of \mathcal{C} . Moreover, if a Sylow p -subgroup $P \leq H$ acts nontrivially on $\mathcal{C}(x, y)$, then it acts nontrivially on $\mathcal{G}(x, y)$ as well. This conclusion comes from the fact that $O^{p'}H \leq H_0$ if and only if $O^{p'}H \leq H_1$, see the remark before this lemma and the equivalent conditions introduced at the beginning of this subsection.

Proposition 6.12. *Suppose that $\text{char}(k) = p \geq 5$. If G (resp., H) has a p -subgroup P (resp., Q) such that $1 \neq P \cap G_0 \neq G_0$ (resp., $1 \neq Q \cap H_0 \neq H_0$), then \mathcal{C} is of infinite representation type.*

Proof. By the reduction before Lemma 6.11 we know $O^{p'}G \leq G_0$. Therefore, we only consider the case that a p -subgroup $Q \leq H$ such that $Q \cap H_0 \neq 1$ and $Q \not\leq H_0$, i.e., $Q\alpha \neq \{\alpha\}$ and $\text{Stab}_Q(\alpha) \neq 1$. Moreover, since p does not divide

$|H_1/H_0| = |G_1/G_0|$, we conclude that $1 \neq Q \cap H_0 = Q \cap H_1$ and $Q \not\leq H_1$. Consequently, the quotient category \mathcal{G} defined above satisfies the given condition as well since $\text{Stab}_H(\bar{\alpha}) = H_1$. Therefore, it suffices to show the infinite representation type of \mathcal{G} . Observe that p divides both $|H_1|$ and $|H : H_1|$, so the Scott module N of $k \uparrow_{H_1}^H$ is neither isomorphic to k nor projective. We have two cases:

Case I: $\text{Soc}(N)$ contains a simple summand $S \not\cong k$. Since $\text{Soc}(N) \cong \text{Top}(N)$ (see [20]), $\text{Top}(N)$ contains a summand isomorphic to S . Thus

$$0 \neq \text{Hom}_{kH}(N, S) \subseteq \text{Hom}_{kH}(k \uparrow_{H_1}^H, S) \cong \text{Hom}_{kH_1}(k, S \downarrow_{H_1}^H),$$

we conclude that $\text{Soc}(S \downarrow_{H_1}^H)$ contains a summand k_1 isomorphic to k .

Write $N = k \oplus S \oplus N' = k \oplus (k_1 \oplus S') \oplus N'$ as vector spaces, where both k and S are simple summands in $\text{Soc}(N)$. Note that N is indecomposable, $k \not\cong S$, and they both are contained in $\text{Soc}(N)$. Therefore, every non-invertible endomorphism on N maps $k \oplus S$ to 0.

Construct a class of representations R_d of \mathcal{G} in the following way, $0 \neq d \in k$. First, $R_d(x) = k$, $R_d(y) = N$. The map $R_d(\alpha)$ is defined by

$$k \xrightarrow{\begin{bmatrix} 1 \\ d \\ 0 \\ 0 \end{bmatrix}} k \oplus (k_1 \oplus S') \oplus M'.$$

By Proposition 6.1, this determines a representation of \mathcal{G} . It is indecomposable. Moreover, if $R_b \cong R_d$, we have the following matrix identity:

$$\begin{bmatrix} 1 \\ d \\ 0 \\ 0 \end{bmatrix} t = \begin{bmatrix} u & 0 & 0 & \varphi_{41} \\ 0 & u & 0 & \varphi_{42} \\ 0 & 0 & u & \varphi_{43} \\ 0 & 0 & 0 & \varphi_{44} \end{bmatrix} \begin{bmatrix} 1 \\ b \\ 0 \\ 0 \end{bmatrix}.$$

In the above identity, $b, d, u \in k$ are nonzero scalars, and $\varphi_{44} = uI + \delta$, where I is the identity matrix and δ is a nilpotent matrix. By computation, we get $b = d$. In this way we construct infinitely many pairwise non-isomorphic indecomposable representations of \mathcal{G} .

Case II: $\text{Soc}(N) \cong \text{Top}(N) \cong k$. In this situation N is a proper quotient module of the projective kH -module P_k . In particular, the multiplicity $[P_k, k] > [N : k] \geq 2$. Therefore, in the Brauer graph Γ of the principal block $B_0(kH)$, there is an exceptional vertex to which the edge k is adjacent.

Let m be the multiplicity of this exceptional vertex and e be the number of edges in Γ . Then $e \mid (p-1)$ and $em = |D| - 1$, where the defect group D of $B_0(kH)$ is a Sylow p -subgroup of kH (see [1, 4]). By the given condition, $|D| \geq p^2$. Therefore, $m \geq p+1$, so P_k has at least $p+2$ composition factors isomorphic to k . Consequently, $\dim_k \text{End}_{kH}(P_k) \geq p+2 \geq 7$. The conclusion follows from Lemma 6.11 (note that $\text{Stab}_H(\bar{\alpha})$ is H_1 instead of H_0 for \mathcal{G}). \square

Actually, we have shown the following conclusion in the above proof:

Corollary 6.13. *Suppose that $\text{char}(k) = p \neq 2, 3$. If \mathcal{C} is of finite representation type, then the Scott module of $k \uparrow_{H_0}^H$ (or $k \uparrow_{H_1}^H$) is either k or projective.*

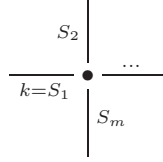
Proof. This is clearly true for $p = 0$. Thus we assume $p \geq 5$. Since p does not divide $|H_1 : H_0|$ by the assumption we made before, the Scott module of $k \uparrow_{H_0}^H$ is isomorphic to that of $k \uparrow_{H_1}^H$, see [7, 13, 20]. By the above proposition, H_0 either contains a Sylow p -subgroup of H or only has a trivial p -subgroup. In the first case the Scott module is isomorphic to k , and in the second case it is isomorphic to the uniserial projective module P_k . \square

We now consider the situation that H has a normal Sylow p -subgroup P . A very special case is that H is the semi-direct product of P and H_0 .

Lemma 6.14. *Suppose that $H \cong P \rtimes H_0$ and $G = 1$, where P is a cyclic p -group and p does not divide $|H_0|$. Then \mathcal{C} is of infinite representation type whenever $\text{char}(k) = p \geq 17$.*

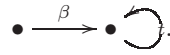
Proof. We prove the conclusion by constructing infinitely many distinct indecomposable representations. Since p does not divide $|H_0|$, $k \uparrow_{H_0}^H \cong P_k$ is the projective cover of the kH -module k . Moreover, $k\mathcal{C}(x, y) = kH\alpha \cong k \uparrow_{H_0}^H \cong P_k$ as kH -modules.

Since $P \triangleleft G$, the Brauer graph Γ of the principal block $B_0(kH)$ is a star (see [1]). Let e be the numbers of edges (simple B_0 -modules) in Γ and m be the multiplicity of the exceptional vertex. Then Γ can be pictured as:



Moreover, we know $em = |P| - 1 \geq p - 1 \geq 16$.

If $m \geq 3$, then k as a composition factor of P_k appears $m + 1 \geq 4$ times, so $d = \text{End}_{kH}(P_k) \geq 4$. Let e_k be a primitive idempotent of kH such that $kHe_k \cong kCe_k \cong P_k$, and define $\Lambda = \text{End}_{k\mathcal{C}}(k\mathcal{C}1_x \oplus kCe_k)^{\text{op}}$. Note that $kH\alpha \cong P_k$. Using the same technique as in the proof of Lemma 6.11, we find that Λ is isomorphic to the path algebra of the following quiver with relation $t^d = 0$:



It is of infinite representation type for $d \geq 4$ by Bongartz's list in [3, 5].

If $m \leq 2$, then $e \geq 8$ and we obtain the following 4 indecomposable kH -modules:

$$\begin{array}{ccccccc}
 & & & & S_{e-2} & & \\
 & & & & S_{e-1} & & \\
 & & & S_e & S_e & & \\
 M_1 = k, & M_2 = k, & M_3 = k, & M_4 = k & & & \\
 & S_2 & S_2 & S_2 & & & \\
 & & S_3 & S_3 & & & \\
 & & & S_4 & & &
 \end{array}$$

We have $\text{Hom}_{kH}(M_i, M_j) = 0$ if $i \neq j$ and $\text{End}_{kH}(M_i) \cong k$, $1 \leq i, j \leq 4$. Moreover,

$$k \cong \text{Hom}_{kH}(P_k, M_i) \cong \text{Hom}_{kH}(k \uparrow_{H_0}^H, M_i) \cong \text{Hom}_{kH_0}(k, M_i \downarrow_{H_0}^H).$$

Therefore, $\text{Soc}(M_i \downarrow_{H_0}^H)$ has a unique simple summand k_i isomorphic to k , $1 \leq i \leq 4$.

Let R be an indecomposable representation of the following quiver Q with dimension vector $(d_1, d_2, d_3, d_4, d_5)$.

$$Q: \begin{array}{ccccc} & & 1 & & \\ & & \uparrow & & \\ 2 & \longleftarrow & 5 & \longrightarrow & 4 \\ & & \downarrow & & \\ & & 3 & & \end{array} \xrightarrow{R} \begin{array}{ccccc} & & k^{d_1} & & \\ & & \uparrow \varphi_1 & & \\ k^{d_2} & \xleftarrow{\varphi_2} & k^{d_5} & \xrightarrow{\varphi_4} & k^{d_4} \\ & & \downarrow \varphi_3 & & \\ & & k^{d_3} & & \end{array}$$

There is a corresponding representation \tilde{R} of \mathcal{C} as follows: $\tilde{R}(x) \cong k^{d_5}$ and $\tilde{R}(y) \cong \bigoplus_{i=1}^4 M_i^{d_i}$. The map $\tilde{R}(\alpha)$ is the composite of the following maps

$$k^{d_5} \xrightarrow{\begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \end{bmatrix}} \bigoplus_{i=1}^4 k_i^{d_i} \xrightarrow{\iota} \bigoplus_{i=1}^4 \text{Soc}(M_i^{d_i} \downarrow_{H_0}^H) \xrightarrow{\epsilon} \bigoplus_{i=1}^4 M_i^{d_i},$$

where ι and ϵ are the inclusion maps. The reader can check that \tilde{R} is indeed a representation of \mathcal{C} (by Proposition 6.1), and is indecomposable. Therefore, we can get infinitely many distinct indecomposable representations of \mathcal{C} . \square

A even more special case in the above lemma is that the indecomposable projective kH -module P_k has only composition factors isomorphic to k , or equivalently, the Brauer graph Γ of the principal block $B_0(kH)$ has only one edge. Let e_k and f be idempotents in kH such that $e_k + f = 1_y$ and $kHe_k \cong P_k$. Then $e_k kHf = f kHe_k = 0$ since k is the only simple module lying in Γ . Note that e_k and f are idempotents of $k\mathcal{C}$ as well, and clearly we have $1_{\mathcal{C}} = e_k + f + 1_x$.

We claim that $k\mathcal{C} = (1_x + e_k)k\mathcal{C}(1_x + e_k) \oplus f k\mathcal{C}f$ is an algebra decomposition. To prove this, we show $(1_x + e_k)k\mathcal{C}f = f k\mathcal{C}(1_x + e_k) = 0$. Since

$$e_k k\mathcal{C}f = e_k kHf = 0 = f kHe_k = f k\mathcal{C}e_k, \quad 1_x k\mathcal{C}f = 0$$

we only need to show $f k\mathcal{C}1_x = 0$. But this is true since

$$f k\mathcal{C}1_x = f kH\alpha \cong \text{Hom}_{kH}(kHf, kH\alpha) \cong \text{Hom}_{kH}(kHf, P_k) \cong f kHe_k = 0.$$

Consider the block $(1_x + e_k)k\mathcal{C}(1_x + e_k)$ of $k\mathcal{C}$. It is isomorphic to the path algebra of the following quiver with relation $\delta^d = 0$, where $d = \dim_k \text{End}_{kH}(P_k) = \dim_k P_k = |P| \geq p \geq 5$.

$$\bullet \xrightarrow{\beta} \bullet \begin{array}{c} \curvearrowright \delta \end{array}.$$

This algebra is of infinite representation type by Bongartz's list, see [3, 5].

Proposition 6.15. *Suppose that $\text{char}(k) = p \geq 17$. If either G or H has a normal Sylow p -subgroup P acting nontrivially on $\mathcal{C}(x, y)$, then \mathcal{C} is of infinite representation type.*

Proof. We can assume $P \triangleleft H$ and want to show that \mathcal{G} , the quotient category of \mathcal{C} defined before Proposition 6.12, is of infinite representation type. Let $H' = PH_1$,

which is a subgroup of H since $P \triangleleft H$. Consider the following subcategory \mathcal{H} of \mathcal{G} :

$$\mathcal{H}: \quad \begin{array}{c} \curvearrowright x \xrightarrow[H']{H'\bar{\alpha}} y \curvearrowright \end{array}$$

Let $\{h_1 = 1, h_2, \dots, h_n\}$ be a chosen set of representatives from the cosets $H \backslash H'$, we can show that ${}_k\mathcal{H}k\mathcal{G}_k\mathcal{H} = k\mathcal{H} \oplus R$ is a decomposition of bimodules, where $R = (\oplus_{i=2}^n kH'h_i) \oplus (\oplus_{i=2}^n kH'h_i\alpha)$. Therefore, it is enough to show the infinite representation type of \mathcal{H} . Factoring out $H_1 \cap P$, a normal subgroup of both H_1 and P , we get a quotient category \mathcal{I} of \mathcal{H} as follows,

$$\mathcal{I}: \quad \begin{array}{c} \curvearrowright x \xrightarrow[\bar{H}]{\bar{H}\bar{\alpha}} y \curvearrowright \end{array}$$

where the quotient group

$$\bar{H} = PH_1/(H_1 \cap P) \cong (P/H_1 \cap P) \rtimes (H_1/H_1 \cap P).$$

Note that $\mathcal{I}(x, y) = \bar{H}\bar{\alpha} = H'\bar{\alpha} = \mathcal{H}(x, y)$ since $P \cap H_1 \leq H_1$ fixes $\bar{\alpha}$. Moreover, the normal Sylow p -subgroup $P/P \cap H_1$ of \bar{H} acts nontrivially on $\mathcal{I}(x, y)$. By the previous lemma we conclude that \mathcal{I} is of infinite representation type, so is \mathcal{H} . The proof is completed. \square

7. A CLASSIFICATION OF REPRESENTATION TYPES FOR SEVERAL CASES.

As in the previous section \mathcal{C} is a connected skeletal finite EI category with two objects x and y such that $\mathcal{C}(y, x) = 0$. Let α , $G_0 \triangleleft G_1 \leq G$ and $H_0 \triangleleft H_1 \leq H$ as defined before. From Theorem 1.1 we find that in many cases the representation type of \mathcal{C} is determined by two pieces of information: the transitivity of the actions by G and H on $\mathcal{C}(x, y)$, and the triviality of the actions by Sylow p -subgroups in G and H . Indeed, the combination of conditions (a-c)

- (a) Both G and H act transitively.
- (b) One of G and H acts transitively, and the other one does not.
- (c) Neither G nor H acts transitively.

and conditions (1-3)

- (1) Both $O^{p'}G$ and $O^{p'}H$ act trivially.
- (2) One of $O^{p'}G$ and $O^{p'}H$ acts trivially, and the other one does not.
- (3) Neither $O^{p'}G$ nor $O^{p'}H$ acts trivially.

give us 8 situations (the combination (a)+(2) cannot happen). By Theorem 1.1, if $\text{char}(k) = p \neq 2, 3$, we get infinite representation type for five cases: (c)+(1), (c)+(2), (c)+(3), (a)+(3), (b)+(3). In the first subsection we will prove the finite representation type for the case (a)+(1). There are still two cases unresolved: (b)+(1) and (b)+(2), which will be studied in the second subsection. In the last subsection we classify the representation type of \mathcal{C} under the extra assumption that both G and H are abelian.

7.1. A family of finite EI categories with finite representation type. We first prove the following lemma.

Lemma 7.1. *If $\mathcal{C}(x, y)$ has only one morphism, then \mathcal{C} is of finite representation type.*

In [10] this result has been proved under the extra assumption that both G and H are abelian and have invertible orders, see Proposition 3.30 in that paper. Actually, if kG and kH are semisimple, we can construct the ordinary quiver \tilde{Q} of $k\mathcal{C}$ using the algorithm described in Section 4. It is not hard to see that \tilde{Q} is the disjoint union of an arrow connecting the trivial kG -module k_G and the trivial kH -module k_H and several isolated vertices. Since this result holds for all algebraically closed fields, we give a uniform proof independent of the characteristic of k .

Proof. Let $p = \text{char}(k)$. Take a Sylow p -subgroup P of G and a Sylow p -subgroup Q of H . This can always be done since if G or H has order invertible in k , we then take the trivial group as the unique Sylow p -subgroup by our convention. Let \mathcal{S} be the following subcategory:

$$\mathcal{S} : \quad \begin{array}{c} \curvearrowright_P \\ x \end{array} \xrightarrow{\alpha} \begin{array}{c} y \\ \curvearrowright_Q \end{array},$$

and define another subcategory \mathcal{J} as below:

$$\mathcal{J} : \quad \begin{array}{c} \curvearrowright_G \\ x \end{array} \xrightarrow{\alpha} \begin{array}{c} y \\ \curvearrowright_Q \end{array}.$$

By Proposition 5.1, \mathcal{S} is of finite representation type. Applying Proposition 6.8 to the pair $(\mathcal{J}, \mathcal{S})$, we conclude that \mathcal{J} is of finite representation type, and so is the opposite category \mathcal{J}^{op} . Applying Proposition 6.8 again to the pair $(\mathcal{C}^{\text{op}}, \mathcal{J}^{\text{op}})$, we obtain the finite representation type of \mathcal{C}^{op} . Therefore, \mathcal{C} is of finite representation type. \square

Now we prove Theorem 1.2.

Theorem 7.2. *Suppose that both G and H act transitively on $\mathcal{C}(x, y)$. If $\text{char}(k) = p \neq 2, 3$, then \mathcal{C} is of finite representation type if and only if all p -subgroups of G and H act trivially on $\mathcal{C}(x, y)$.*

Proof. Since both G and H act transitively on $\mathcal{C}(x, y)$, we have $G_0 \triangleleft G_1 = G$, $H_0 \triangleleft H_1 = H$. If G or H has a p -subgroup acting nontrivially on $\mathcal{C}(x, y)$, so does the other one by the remark before Proposition 6.10. Therefore, \mathcal{C} has infinite representation type by Proposition 6.10.

On the other hand, if all p -groups of G and H act trivially on $\mathcal{C}(x, y)$, then $|G/G_0| = |H/H_0| = n$ is invertible in k . Let \mathcal{K} be the following subcategory:

$$\mathcal{K} : \quad \begin{array}{c} \curvearrowright_{G_0} \\ x \end{array} \xrightarrow{\alpha} \begin{array}{c} y \\ \curvearrowright_{H_0} \end{array},$$

which is of finite representation type by the previous lemma. Applying Proposition 6.8 to the pair $(\mathcal{C}, \mathcal{K})$, we conclude that \mathcal{C} is of finite representation type as well. \square

7.2. Criteria by permutation modules. In this subsection we develop more criteria which can be applied to investigate the cases (b)+(1) and (b)+(2). By previous results, we can assume that H acts transitively on $\mathcal{C}(x, y)$. Therefore, $G_1 = G$. We identify $k(G_1/G_0)$ and $k(H_1/H_0)$. **We do not suppose that $O^{p'}G$ acts trivially on this biset**, so $|H_1 : H_0| = |G_1 : G_0|$ might not be invertible in k , and hence $k \uparrow_{H_0}^{H_1} \cong k(H_1/H_0)$ might not be semisimple. We will find in many situations the representation type of \mathcal{C} is determined by the structure of induced modules $S \uparrow_{H_1}^H$, where S is a summand of $\text{Top}(k \uparrow_{H_0}^{H_1})$.

The following lemma will be used frequently.

Lemma 7.3. *Let M be a kH -module and S be a simple summand of $\text{Top}(k \uparrow_{H_0}^{H_1})$. Then $\text{Hom}_{kH}(M, S \uparrow_{H_1}^H) \neq 0$ if and only if S is a summand of $\text{Top}(M \downarrow_{H_1}^H)$. Dually, $\text{Hom}_{kH}(S \uparrow_{H_1}^H, M) \neq 0$ if and only if S is a summand of $\text{Soc}(M \downarrow_{H_1}^H)$.*

Proof. Use Frobenius reciprocity. \square

The following proposition generalizes Corollary 6.5 in [14], where kH is supposed to be semisimple.

Proposition 7.4. *If \mathcal{C} is of finite representation type, then for every simple summand S of $\text{Top}(k \uparrow_{H_0}^{H_1})$, the following conditions must be true:*

- (1) $\text{Top}(S \uparrow_{H_1}^H)$ has no repeated summands;
- (2) $\text{Top}(S \uparrow_{H_1}^H)$ has at most 3 summands;
- (3) every indecomposable summand M of $S \uparrow_{H_1}^H$ is uniserial or biserial, and if M is biserial, it is projective;
- (4) if $M \not\cong N$ are indecomposable summands of $S \uparrow_{H_1}^H$, then $\text{Hom}_{kH}(M, N) = 0$.

Proof. Suppose that a simple kH -module U appears at least twice in $\text{Top}(S \uparrow_{H_1}^H)$. Therefore, by Frobenius reciprocity S appears at least twice in $\text{Soc}(U \downarrow_{H_1}^H)$. Take two summands $S_1 \cong S \cong S_2$ in $\text{Top}(U \downarrow_{H_1}^H)$ and write $U = S_1 \oplus S_2 \oplus U'$ as vector spaces. Define a family of representations R_d for \mathcal{C} as follows, $0 \neq d \in k$:

$$\varphi_d : S \xrightarrow{\begin{bmatrix} 1 \\ d \\ 0 \end{bmatrix}} S_1 \oplus S_2 \oplus U'.$$

Note that we regard S on the left side as a simple $k(G/G_0)$ -module (so it is also a simple kG -module). As we did in the proof of Proposition 6.12, we can check that R_d is indecomposable, and that $R_d \cong R_b$ if and only if $b = d$. Thus (1) must be true.

Suppose that $\text{Top}(S \uparrow_{H_1}^H)$ has more than 3 summands. By (1) we can assume that all of them are non-isomorphic. Take four summands U_i , $1 \leq i \leq 4$. Again by Frobenius reciprocity we know that S appears exactly once in $\text{Soc}(U_i \downarrow_{H_1}^H)$. Write $U_i = S_i \oplus U'_i$, $1 \leq i \leq 4$. Then as we did in the proof of Lemma 6.14, we can obtain infinitely many pairwise non-isomorphic indecomposable representations for \mathcal{C} . Thus (2) must be true.

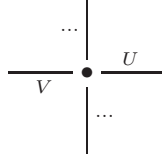
We show that every indecomposable summand M of $S \uparrow_{H_1}^H$ has a simple top. This implies (3). Indeed, since M has a simple top, it is a quotient module of an indecomposable projective kH -module, and hence must be uniserial or biserial (see [1]). Because $k \uparrow_{H_1}^H$ is isomorphic to its dual module, M must have a simple socle. Therefore, if M is biserial, it must be isomorphic to its projective cover.

Suppose that $\text{Top}(M)$ contains more than one simple summands. Take U and V from $\text{Top}(M)$. By (1), U is not isomorphic to V . By the previous lemma and (1), $U \downarrow_{H_1}^H$ contains exactly a simple summand $S_1 \cong S$. Similarly, $V \downarrow_{H_1}^H$ contains exactly a simple summand $S_2 \cong S$ as well.

Let ΩM be the first syzygy of M . Then $\Omega M \neq 0$ since M is indecomposable and its top is not simple. Moreover, U and V appears as summands of $\text{Soc}(\Omega M)$. Therefore, as we did in the proof of (1), using ΩM we can construct a family of

infinitely many non-isomorphic indecomposable representations of \mathcal{C} . This contradiction shows that M must have a simple top, and (3) is proved.

Now we turn to (4). Suppose that $\text{Hom}_{kH}(M, N) \neq 0$. By (3) we can suppose that M (resp., N) is a quotient module of an indecomposable projective module P_U (resp., P_V). Therefore, if $\text{Hom}_{kH}(M, N) \neq 0$, the multiplicity $[N : U] \neq 0$, so $[P_V, U] \neq 0$. This happens if and only if U and V lie in the same block B of kH , and U and V are connected to the same vertex in the Brauer graph Γ of B . Therefore, Γ has a piece as follows:



So we can define a uniserial kH -module L such that $\text{Top}(L) \cong \text{Top}(N) \cong V$, $\text{Soc}(L) \cong \text{Soc}(M) \cong U$, and $[L : U] = [L : V] = 1$. Therefore, L is a quotient module of P_V , and it is actually a quotient module of N .

Clearly, $\text{Hom}_{kH}(N, L) \neq 0 \neq \text{Hom}_{kH}(M, L)$. Therefore,

$$\dim_k \text{Hom}_{kH_1}(S, L \downarrow_{H_1}^H) = \dim_k \text{Hom}_{kH}(S \uparrow_{H_1}^H, L) \geq 2.$$

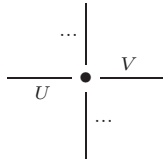
So we can find two simple summands $S_1 \cong S \cong S_2$ in $\text{Soc}(L \downarrow_{H_1}^H)$. Note that $\text{End}_{kH}(N) \cong k$. Using this module L , as we did in the proof of (1) we can construct infinitely many indecomposable non-isomorphic representations of \mathcal{C} . The conclusion of (4) follows from this contradiction. \square

In the case that kH_1 is semisimple, the conclusion (4) in this proposition can be strengthened.

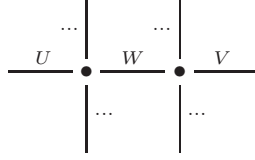
Proposition 7.5. *Let S be a simple summand of $\text{Top}(k \uparrow_{H_0}^{H_1})$ and suppose that \mathcal{C} is of finite representation type. If kH_1 is semisimple, then distinct summands of $S \uparrow_{H_1}^H$ lie in different blocks of kH . Moreover, if $\text{char}(k) = p \geq 5$ and p does not divide $|H_1|$ but divides $|H|$, then H_1 is not a normal subgroup of H .*

Proof. Since kH_1 is semisimple, $S \uparrow_{H_1}^H$ is projective. If there are two indecomposable summands $P_U \not\cong P_V$ of $S \uparrow_{H_1}^H$ such that the simple kH -modules U and V lie in the same block B of kH , then we can construct infinitely many indecomposable non-isomorphic representations of \mathcal{C} as follows.

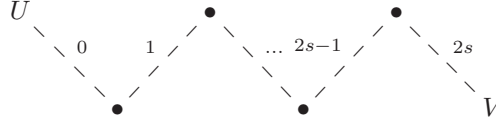
Let Γ be the Brauer graph of B . Since Γ contains two edges U and V , the defect group D of B is nontrivial. Let l be the length of the shortest path (excluding the edges U and V) connecting U and V in Γ . For example, if $l = 0$, then Γ has a piece as follows:



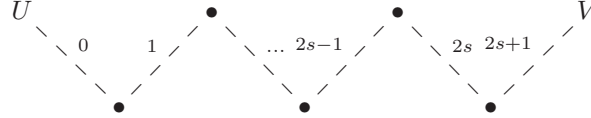
Then we can find an indecomposable kH -module M such that $\text{Soc}(M) \cong U$ and $\text{Top}(M) \cong V$. if $l = 1$, then Γ has a piece as follows:



Then we can find an indecomposable kH -module M such that $\text{Soc}(M) \cong U \oplus V$ and $\text{Top}(M) \cong W$. In general, if $l = 2s$ is even, we can find an indecomposable kH -module M with the following structure:



where each dotted line segment is a uniserial kH -module. Similarly, if $l = 2s + 1$ is odd, we can find an indecomposable kH -module with the following structure:



In all cases, we can get an indecomposable kH -module M containing composition factors U and V .

By Lemma 7.3, both $U \downarrow_{H_1}^H$ and $V \downarrow_{H_1}^H$ contain composition factors isomorphic to S . Note that kH_1 is semisimple. Therefore, we get a kH_1 -module decomposition

$$M \downarrow_{H_1}^H \cong U \downarrow_{H_1}^H \oplus V \downarrow_{H_1}^H \oplus M' \downarrow_{H_1}^H \cong (S_1 \oplus U') \oplus (S_2 \oplus V') \oplus M' \cong S_1 \oplus S_2 \oplus M_1,$$

where $S_1 \cong S \cong S_2$.

As in the proof of (1) of the previous proposition we can define a family of representations R_d for \mathcal{C} as follows, $0 \neq d \in k$:

$$\varphi_d : S \xrightarrow{\begin{bmatrix} 1 \\ d \\ 0 \end{bmatrix}} S_1 \oplus S_2 \oplus M_1.$$

They are indecomposable and pairwise non-isomorphic if $b \neq d$. Thus \mathcal{C} is of infinite representation type. This contradiction tells us that different summands of $k \uparrow_{H_1}^H$ must be in distinct blocks, and the first statement is proved.

Now suppose that $p \geq 5$ and $|H_1|$ is invertible in k . If $H_1 \triangleleft H$, by factoring out this normal subgroup and G , we get a quotient category \mathcal{M} of \mathcal{C} as follows

$$\mathcal{M} : \quad \begin{array}{ccc} \circlearrowleft & \xrightarrow{\bar{H}\bar{\alpha}} & \circlearrowright \\ x & \xrightarrow{\dots} & y \end{array}$$

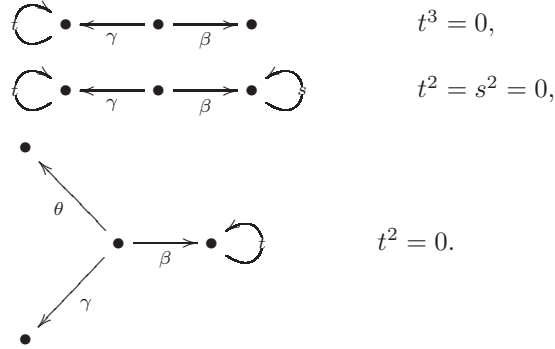
where $\bar{H} = H/H_1$ acts regularly on this biset. Clearly, \bar{H} has a nontrivial Sylow p -subgroup P . Define \mathcal{N} to be the following category:

$$\mathcal{N} : \quad \begin{array}{ccc} \circlearrowleft & \xrightarrow{P\bar{\alpha}} & \circlearrowright \\ x & \xrightarrow{\dots} & y \end{array}$$

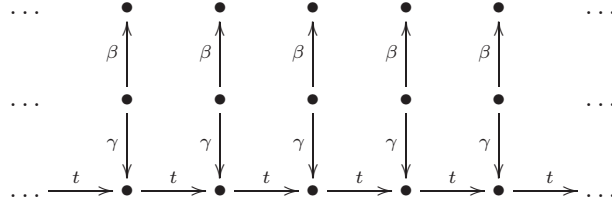
As in the proof of Proposition 6.15, we can show $k\mathcal{N} \mid ({}_k\mathcal{N}k\mathcal{M}_{k\mathcal{N}})$. Since \mathcal{N} is of infinite representation type by Proposition 5.1, so are \mathcal{M} and \mathcal{C} . This contradiction tells us that H_1 cannot be a normal subgroup of H . \square

In the following lemma we give some algebras of infinite representation types. They will be used in the proof of the next proposition.

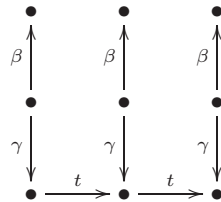
Lemma 7.6. *The path algebras of the following quivers with relations are of infinite representation type:*



Proof. These results can be proved by the covering theory. As example, we show the first algebra is of infinite representation type, and leave the other two to the reader. A universal covering of the first quiver is described as follows with relation $t^3 = 0$:



This covering is not locally representation-finite since it contains a subquiver of infinite representation type as below:



Therefore, the first algebra is of infinite representation type. \square

Proposition 7.7. *If \mathcal{C} is of finite representation type, then $\dim_k \text{End}_{kH}(k \uparrow_{H_1}^H) \leq 3$. That is, $H_1 \backslash H / H_1$ has at most three double cosets.*

The second statement comes from

$$\begin{aligned}
\mathrm{Hom}_{kH}(k \uparrow_{H_1}^H, k \uparrow_{H_1}^H) &\cong \mathrm{Hom}_{kH_1}(k \uparrow_{H_1}^H \downarrow_{H_1}^H, k) \\
&\cong \bigoplus_{s \in H_1 \backslash H / H_1} \mathrm{Hom}_{kH_1}((s \otimes k) \uparrow_{H_1 \cap sH_1 s^{-1}}^{H_1}, k) \\
&\cong \bigoplus_{s \in H_1 \backslash H / H_1} \mathrm{Hom}_{k(H_1 \cap sH_1 s^{-1})}(s \otimes k, k).
\end{aligned}$$

So $\dim_k \mathrm{End}_{kH}(k \uparrow_{H_1}^H)$ coincide with the number of double cosets of $H_1 \backslash H / H_1$.

Proof. Note that \mathcal{C} has a quotient category \mathcal{G} (defined before Proposition 6.12) such that $\mathcal{G}(x, x) = 1$, $\mathcal{G}(y, y) = H$, and $\mathcal{C}(x, y) = H\bar{\alpha}$ where $\mathrm{Stab}_H(\bar{\alpha}) = H_1$. Therefore, it suffices to show that \mathcal{G} is of infinite representation type if $\dim_k \mathrm{End}_{kH}(k \uparrow_{H_1}^H) \geq 4$.

Denote all indecomposable summands of $k \uparrow_{H_1}^H$ by M_1, M_2, \dots, M_n . By Proposition 7.4, each of them is a quotient module of an indecomposable projective module of kH . Let e_1, e_2, \dots, e_n be primitive idempotent elements in kH such that $P_i = kHe_i$ is a projective cover of M_i , $1 \leq i \leq n$. By Proposition 7.4, we can assume the following properties: $n \leq 3$; $\mathrm{Hom}_{kH}(P_i, M_j) = 0$, $1 \leq i \neq j \leq n$. Consider the algebra

$$\Lambda = \mathrm{End}_{k\mathcal{G}}(k\mathcal{G}(1_x + e_1 \dots + e_n))^{\mathrm{op}} \cong (e_1 + \dots + e_n + 1_x)k\mathcal{G}(e_1 + \dots + e_n + 1_x).$$

As in the proof of Lemma 6.10, we only need to show that Λ is of infinite representation type if $\dim_k \mathrm{End}_{kH}(k \uparrow_{H_1}^H) \geq 4$ since by Proposition 2.5 on page 36 in [2] $\Lambda\text{-mod}$ is equivalent to a subcategory of $k\mathcal{G}\text{-mod}$.

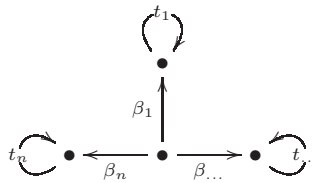
Observe that $1_x k\mathcal{G}(e_1 + \dots + e_n) = 0$, $1_x k\mathcal{G}1_x \cong k$, and

$$\begin{aligned}
(e_1 + \dots + e_n)k\mathcal{G}1_x &= (e_1 + \dots + e_n)kH\bar{\alpha} \\
&\cong \mathrm{Hom}_{kH}(kH(e_1 + \dots + e_n), kH\bar{\alpha}) \\
&\cong \mathrm{Hom}_{kH}(kH(e_1 + \dots + e_n), k \uparrow_{H_1}^H) \\
&\cong \mathrm{Hom}_{kH}(kH(e_1 + \dots + e_n), M_1 \oplus \dots \oplus M_n) \\
&\cong \mathrm{Hom}_{kH}(P_1 + \dots + P_n, M_1 \oplus \dots \oplus M_n) \\
&\cong \bigoplus_{i=1}^n \mathrm{Hom}_{kH}(P_i, M_i)
\end{aligned}$$

as left $(e_1 + \dots + e_n)k\mathcal{G}(e_1 + \dots + e_n)$ -modules since $\mathrm{Hom}_{k\mathcal{G}}(P_i, M_j) = 0$ for $1 \leq i \neq j \leq n$. By the same reasoning, the product of an element in the two-sided Λ -ideal generated by $\bigoplus_{1 \leq i \neq j \leq n} \mathrm{Hom}_{kH}(P_i, P_j)$ and an element in $(e_1 + \dots + e_n)k\mathcal{G}1_x$ is 0. We also have:

$$\dim_k \mathrm{Hom}_{kH}(kHe_i, M_i) = \dim_k \mathrm{End}_{kH}(M_i).$$

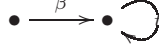
Therefore, modulo the two-sided Λ -ideal generated by $\bigoplus_{1 \leq i \neq j \leq n} \mathrm{Hom}_{kH}(P_i, P_j)$, Λ has a quotient algebra $\bar{\Lambda}$ which is isomorphic to the path algebra of the following bounded quiver



with relations $t_i^{a_i} = 0 = t_i^{d_i} \beta_i$, where $a_i \geq d_i = \dim_k \text{End}_{kH}(M_i)$, $1 \leq i \leq n$.

Suppose that $d = \dim_k \text{End}_{kH}(k \uparrow_{H_1}^H) \geq 4$. Note that $d = \sum_{i=1}^n d_i$. We have several cases:

Case I: $n = 1$. Therefore, $k \uparrow_{H_1}^H$ is an indecomposable kH -module, and $\bar{\Lambda}$ is isomorphic to the path algebra of the following quiver with relation $t^a = 0 = t^d \beta$, $a \geq d$. we deduce that $\bar{\Lambda}$ is of infinite representation type from Bongartz's list in [3, 5] if $d \geq 4$.



Case II: $n = 2$, so $d_1 + d_2 = d \geq 4$. If one of them is 1, then $\bar{\Lambda}$ has the first algebra in the previous lemma as a quotient algebra; if both numbers are at least 2, then $\bar{\Lambda}$ has the second algebra in the previous lemma as a quotient algebra.

Case III: $n = 3$, so $d_1 + d_2 + d_3 \geq 4$. Therefore, at least one of them must be bigger than 1, and $\bar{\Lambda}$ has the third algebra in the previous lemma as a quotient algebra.

Consequently, in all cases we get infinite representation type for $\bar{\Lambda}$. This contradiction tells us that $d \leq 3$. \square

7.3. Representation types of finite EI categories with abelian automorphism groups. Throughout this subsection we assume that G and H are abelian, and $\text{char}(k) = p \neq 2, 3$. As before, we suppose that H acts transitively on $\mathcal{C}(x, y)$. Note that when $p = 0$, by our convention 1 is the unique p -subgroup of a finite group.

We first describe some easy observation.

Lemma 7.8. *Suppose that \mathcal{C} is of finite representation type and $\text{char}(k) = p \neq 2, 3$. Then both $Op'G$ and $Op'H$ act trivially on $\mathcal{C}(x, y)$, and $|H : H_1| \leq 3$.*

Proof. Since $\text{char}(p) = p \neq 2, 3$, we know that $Op'G$ must act trivially on $\mathcal{C}(x, y)$. Let P be the unique Sylow p -subgroup of H . Without loss of generality we assume that $P \neq 1$ (so $p \geq 5$). By Proposition 6.13, either $P \leq H_0 \leq H_1$ or $P \cap H_0 = 1$. We show the second possibility cannot happen. Indeed, if $P \cap H_0 = 1$, then $P \cap H_1 = 1$ as well (see the remark before Lemma 6.11). But by the last statement of Proposition 7.5, H_1 cannot be a normal subgroup of H . This contradicts the condition that H is abelian. Consequently, $P \leq H_0$, and the first statement is proved.

Now we assume $P \leq H_0 \leq H_1$. Factoring out H_1 and G_1 , we get a quotient category \mathcal{G} as defined before Proposition 6.12 such that $\mathcal{G}(x, x) = 1$ and $\mathcal{G}(y, y) = H/H_1 = \bar{H}$ acts regularly on $\mathcal{G}(x, y)$. Applying Proposition 7.7 to \mathcal{G} we conclude $|H : H_1| = |H_1 \backslash H/H_1| \leq 3$. \square

Therefore, if \mathcal{C} is of finite representation type, $H \cong H_1 \times C_n$ where C_n is a cyclic group of order n , $1 \leq n \leq 3$.

From now on we suppose that $Op'G \leq G_0$ and $Op'H \leq H_0$. Note that $k(G/G_0) = k(G_1/G_0)$ is a semisimple kG -module with pairwise non-isomorphic simple summands. Let S_1, \dots, S_r be all simple summands. Correspondingly, let e_1, \dots, e_r be primitive idempotents in kG such that kGe_i is a projective cover of S_i , $1 \leq i \leq r$. Denote $1 - e_1 - \dots - e_r$ by f . These simple modules S_i can be viewed as kH_1 -modules as well since $G/G_0 \cong H_1/H_0$, and we let E_i be idempotents in kH such that kHE_i is the projective cover of $kH \otimes_{kH_1} S_i$, $1 \leq i \leq r$. By the previous lemma, $H \cong H_1 \times C_n$ with $n \leq 3$, so $kH \otimes_{kH_1} S_i$ has n simple summands, and each E_i is

a sum of n primitive idempotents in kH . Let $F = 1 - E_1 - \dots - E_r$. Note that

$$kH\alpha \cong k \uparrow_{H_0}^H \cong kH \otimes_{kH_1} (k \uparrow_{H_0}^{H_1}) \cong \oplus_{i=1}^r kH \otimes_{kH_1} S_i.$$

Lemma 7.9. *Notation as above. Then $k\mathcal{C} = \oplus_{i=1}^r \Lambda_i \oplus kHF \oplus kGf$ as algebras, where $\Lambda_i = kHE_i \oplus kGe_i \oplus kHE_i\alpha$ as vector spaces.*

Proof. Since $\mathcal{C}(x, y) = H\alpha$, $1_y = F + E_1 + \dots + E_r$, and

$$KHF\alpha \cong \oplus_{i=1}^r F(kH \otimes_{kH_1} S_i) = 0$$

since $kH \otimes_{kH_1} S_i$ has a projective cover kHE_i . Therefore, the first decomposition is true as vector spaces, and we want to show that all summands are $k\mathcal{C}$ -ideals. Particularly, it suffices to show the following identities: $kHE_i\alpha G \subseteq kHE_i\alpha$ and $kH\alpha Ge_i \subseteq kHE_i\alpha$ for $1 \leq i \leq r$; $kHF\alpha G = 0$; and $kH\alpha Gf = 0$.

First, we have

$$kH\alpha Gf = kH\alpha f = \oplus_{i=1}^r kH \otimes_{kH_1} S_i f = 0$$

since $f = 1 - \sum_{i=1}^r e_i$ and kGe_i is the projective cover of S_i .

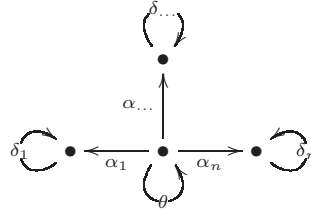
The identity $KHF\alpha G = 0$ is clear as we have shown $kHF\alpha = 0$. The identity $kHE_i\alpha G \subseteq kHE_i\alpha$ is also clear since $\alpha G \subseteq H\alpha$ and H is abelian. We claim $kH\alpha e_i = kHE_i\alpha = kHE_i\alpha e_i$, implying $kH\alpha Ge_i = kH\alpha e_i \subseteq kHE_i\alpha$. Indeed,

$$\begin{aligned} E_i kH\alpha(1 - e_i) &\cong \oplus_{j=1}^r E_i(kH \otimes_{kH_1} S_j)(1 - e_i) \\ &\cong E_i(kH \otimes_{kH_1} (S_1 \oplus \dots \oplus S_{i-1} \oplus S_{i+1} \oplus \dots \oplus S_n)) \\ &\cong E_i(S_1 \uparrow_{H_1}^H \oplus \dots \oplus S_{i-1} \uparrow_{H_1}^H \oplus S_{i+1} \uparrow_{H_1}^H \oplus \dots \oplus S_n \uparrow_{H_1}^H) \end{aligned}$$

is 0 since kHE_i is the projective cover of $S_i \uparrow_{H_1}^H$, so $kHE_i\alpha = kHE_i\alpha e_i$. Similarly, we can show $(1 - E_i)kH\alpha e_i = 0$. Therefore, $kH\alpha e_i = kHE_i\alpha e_i$. \square

We see at once that \mathcal{C} is of finite representation type if and only if all Λ_i are of finite representation type, $1 \leq i \leq r$, for which the structure is described by the following lemma.

Lemma 7.10. *Let Λ_i be as in the previous lemma, $1 \leq i \leq r$. Then it is isomorphic to the path algebra of the following bounded quiver with relations: $\delta_i^t = \delta_i\alpha_i = \alpha_i\theta = \theta^s = 0$, $1 \leq i \leq n = |H : H_1|$, where $t = |Op' H|$ and $s = |Op' G|$.*



Proof. Note that $H \cong H_1 \times C_n$. Therefore, $S_i \uparrow_{H_1}^H$ is semisimple and has n non-isomorphic simple summands, and $kHE_i = E_i kHE_i$ is a direct sum of endomorphism algebras of n non-isomorphic projective kH -modules. Consequently, $kHE_i \cong \oplus_{i=1}^n k[\delta_i]/(\delta_i^t)$ where $t = |Op' H|$. Similarly, $kGe_i \cong e_i kGe_i$ is the endomorphism algebra of an indecomposable projective kG -module. Therefore, $kGe_i \cong k[\theta]/(\theta^s)$ where $s = |Op' G|$.

As a left $E_i k H E_i$ -module, we have

$$\begin{aligned} E_i k H \alpha &\cong \text{Hom}_{kH}(k H E_i, k H \alpha) \cong \text{Hom}_{kH}(k H E_i, \oplus_{j=1}^r (S_j \uparrow_{H_1}^H)) \\ &\cong \text{Hom}_{kH}(k H E_i, S_i \uparrow_{H_1}^H) \end{aligned}$$

which is the space of all homomorphisms from $k H E_i$ to its top $S_i \uparrow_{H_1}^H$. Since $k H E_i \alpha = k H E_i \alpha e_i$ from the proof of the previous lemma, as an $e_i k G e_i$ -module,

$$\begin{aligned} E_i k H \alpha e_i &\cong \oplus_{j=1}^r E_i (k H \otimes_{k H_1} S_j) e_i \cong (k H \otimes_{k H_1} S_i) e_i \\ &\cong (S_i e_i)^d \cong \text{Hom}_{e_i k G e_i}(e_i k G, S_i)^d. \end{aligned}$$

which is the direct sum of d copies of projections from $e_i k G$ to its top S_i , where $d = |H : H_1|$. The conclusion follows from these observations. \square

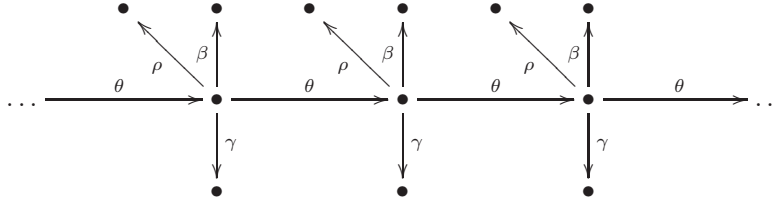
Now we are ready to classify the representation types of finite EI categories with two objects whose automorphism groups are abelian for $p \neq 2, 3$.

Theorem 7.11. *Suppose that both G and H are abelian, $\text{char}(k) = p \neq 2, 3$. Without loss of generality assume that H acts transitively on $\mathcal{C}(x, y)$. Let $s = |O^{p'} G|$, $t = |O^{p'} H|$, and $n = |H : H_1|$. Then \mathcal{C} is of finite representation type if and only if both $O^{p'} G$ and $O^{p'} H$ act trivially on $\mathcal{C}(x, y)$, and one of the following conditions hold:*

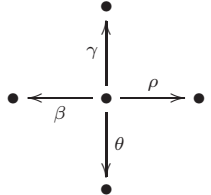
- (1) $n = 1$ for $s, t \geq 5$;
- (2) $n \leq 2$ for $s = 1, t \geq p$ or $t = 1, s \geq p$;
- (3) $n \leq 3$ for $t = s = 1$.

Proof. By Lemma 7.8, we know that n cannot be bigger than 3 for all cases. Lemmas 7.9 and 7.10 tell us that it suffices to consider the representation types of path algebras described in Lemma 7.10. Now we can use covering theory, Auslander-Reiten quivers, and Proposition 5.1 to get the conclusion.

As an example, let us show the case $t = 1$, $s \geq p \geq 5$, and $n = 3$ is of infinite representation type. In that case the corresponding path algebra has the following covering:



with relations $\theta^s = \rho\theta = \beta\theta = \gamma\theta = 0$. This covering is not locally representation-finite since it has a subquiver as



Therefore, \mathcal{C} has infinite representation type for $t = 1$, $s \geq 5$ and $n = 3$. \square

8. THE GENERAL CASE

We already obtain several criteria for finite EI categories satisfying some extra properties in previous sections. Of course, these conditions are not sufficient to guarantee finite representation type of arbitrary finite EI categories. Thus we have to derive more criteria. We focus on finite free EI categories since by Proposition 2.3 every finite EI category is quotient category of its free EI cover, and the finite representation type of this finite free EI cover implies the finite representation type of the original category. Throughout this section let \mathcal{C} be a connected skeletal finite free EI category such that the conclusion of Theorem 1.1 holds for every pair of distinct objects $x, y \in \text{Ob } \mathcal{C}$. Otherwise, we can conclude that \mathcal{C} is of infinite representation type.

Let $Q = (Q_0, Q_1)$ be the underlying quiver of \mathcal{C} (see the end of Section 2 for the definition). By abuse of notation, we identify Q with its free category (see [18] for the definition of free category of a quiver) which is a finite EI category as well. By the definition, $Q_0 = \text{Ob } \mathcal{C}$, and Q_1 is precisely a chosen set of representative unfactorizable morphisms in \mathcal{C} . Define a functor F from \mathcal{C} to Q as follows: F is the identity map restricted to objects. It sends every automorphism $\delta \in \mathcal{C}(x, x)$ to 1_x for $x \in \text{Ob } \mathcal{C}$. Let α be a non-isomorphism in \mathcal{C} . Then it can be written as a composite of several representative unfactorizable morphisms $\alpha_n : x_{n-1} \rightarrow x_n, \dots, \alpha_2 : x_1 \rightarrow x_2$ and some automorphisms in \mathcal{C} . The image of α is defined as $\alpha_n \circ \dots \circ \alpha_2$, a morphism in Q . The reader can use the unique factorization property of finite free EI categories in Definition 2.4 to check that F is well defined. It is clearly a full functor. As a result, Q is a quotient category of \mathcal{C} , and kQ is a quotient algebra of $k\mathcal{C}$. So the infinite representation type of Q implies the infinite representation type of \mathcal{C} . We have actually proved the following result, which is the first condition in Theorem 1.4.

Proposition 8.1. *Let \mathcal{C} be a connected skeletal finite free EI category. Then its underlying quiver Q and underlying poset \mathcal{P} coincide. If \mathcal{C} is of finite representation type, then Q is a Dynkin quiver.*

To determine the representation types of arbitrary finite free EI categories, we have to consider the behavior of objects at which two or more representative unfactorizable morphisms start or end. For $x, y \in \text{Ob } \mathcal{C}$, we say that x is *adjacent to* y if there is an unfactorizable morphism from x to y or from y to x . Clearly, x is adjacent to y if and only if y is adjacent to x . Moreover, the reader can check that it is equivalent to one of the following conditions:

- (1) there is a representative unfactorizable morphism in \mathcal{C} connecting x and y ;
- (2) there is an arrow in the Q connecting x and y .

By the previous proposition, each object in \mathcal{C} has at most three adjacent objects given that \mathcal{C} is of finite representation type. Otherwise, Q is not a Dynkin quiver, so kQ and hence $k\mathcal{C}$ are of infinite representation type.

Lemma 8.2. *Let G be a finite group with cyclic Sylow p -subgroups. Let G_1 and G_2 be two proper subgroups. Suppose that $k \uparrow_{G_1}^G$ and $k \uparrow_{G_2}^G$ has no common composition factors except k . If $|G_1 \setminus G/G_2| \geq 2$, then P_k has only composition factors isomorphic to k , and the Scott modules of both permutation modules are quotient modules of P_k and are not isomorphic to k .*

$$\begin{aligned} \dim_k \operatorname{Hom}_{kG}(k \uparrow_{G_1}^G, k \uparrow_{G_2}^G) &= \dim_k \operatorname{Hom}_{kG_1}(k, k \uparrow_{G_2 \downarrow G_1}^G) \\ &= \dim_k \operatorname{Hom}_{kG_1}(k, \bigoplus_{s \in G_1 \setminus G/G_2} (s \otimes k) \uparrow_{G_1 \cap sG_2s^{-1}}^{G_1}) \end{aligned}$$
$$\mathrm{Hom}_{kG}(k \uparrow_{G_1}^G, k \uparrow_{G_2}^G) = \bigoplus_{i=1}^m \bigoplus_{j=1}^n \mathrm{Hom}_{kG}(M_i, N_j) \cong \bigoplus_{j=1}^n \mathrm{Hom}_{kG}(M_1, N_j).$$
$$\mathcal{D}: \quad \begin{array}{ccccc} \textcircled{G} & \xrightarrow{H\alpha G} & y & \xrightarrow{L\beta H} & z \textcircled{L} \\ \text{...} & & \text{...} & & \\ & & \textcircled{H} & & \end{array} \quad \begin{array}{ccccc} \textcircled{G} & \xrightarrow{H\alpha G} & y & \xleftarrow{H\beta L} & z \textcircled{L} \\ \text{...} & & \text{...} & & \\ & & \textcircled{H} & & \end{array}$$

Proof. We only prove the first statement for the first category since the proof also works for the second one with a very small modification. Let $G_1 = \text{Stab}_G(H\alpha)$ and $L_1 = \text{Stab}_L(\beta H)$. Suppose that H does not act transitively on either $\mathcal{D}(x, y)$ or $\mathcal{D}(y, z)$. By considering the full subcategory \mathcal{E} with objects x and y we conclude that G acts transitively on $\mathcal{D}(x, y)$. Similarly, L acts transitively on $\mathcal{D}(y, z)$. Therefore,

G_1 and L_1 are proper subgroups of G and L respectively. We claim that there is an indecomposable projective kG -module P_S such that $S \not\cong k_G$ and $\text{Top}(P_S \downarrow_{G_1}^G)$ has a simple summand isomorphic to k . Dually, there is an indecomposable projective kL -module P_T such that $T \not\cong k_L$ and $\text{Soc}(P_T \downarrow_{L_1}^L)$ has a simple summand isomorphic to k .

We prove the first claim since the second one is a dual statement. By Lemma 7.3, it is equivalent to saying that $k \uparrow_{G_1}^G$ has a composition factor $S \not\cong k_G$, which is obviously true for $p = 0$. Thus we let $p \geq 5$. If $k \uparrow_{G_1}^G$ has only composition factors isomorphic to k , then it is indecomposable. By Proposition 6.13, either $|G_1|$ is invertible or G_1 contains a Sylow p -subgroup of G . The second case cannot happen. Otherwise $k \uparrow_{G_1}^G \cong k$, which is absurd since G_1 is a proper subgroup of G . Therefore, $|G_1|$ must be invertible in k and $k \uparrow_{G_1}^G \cong P_k$. Moreover, G has a nontrivial Sylow p -subgroup since $\dim_k P_k = |G : G_1| > 1$. Consequently, $\dim_k \text{End}_{kG}(P_k) = \dim_k P_k \geq p \geq 5$, so \mathcal{E} is of infinite representation type by Proposition 7.7. This is not allowed, and the claim is proved.

As before, for an indecomposable representation R of the following quiver with dimension vector $(d_1, d_2, d_3, d_4, d_5)$,

$$\begin{array}{ccccc}
 & 1 & & k^{d_1} & \\
 & \downarrow & & \downarrow \varphi_1 & \\
 2 & \longrightarrow 5 & \longrightarrow 4 & \xRightarrow{R} k^{d_2} & \xrightarrow{\varphi_2} k^{d_5} \xrightarrow{\varphi_4} k^{d_4} \\
 & \downarrow & & & \downarrow \varphi_3 \\
 & 3 & & & k^{d_3}
 \end{array}$$

we can construct a corresponding representation \tilde{R} of \mathcal{D} as follows: $\tilde{R}(x) \cong k^{d_1} \oplus P_S^{d_2}$, $\tilde{R}(y) \cong k^{d_5}$, $\tilde{R}(z) \cong k^{d_3} \oplus P_T^{d_4}$. Linear maps $\tilde{R}(\alpha)$ and $\tilde{R}(\beta)$ can be defined in a way as we did in the proof of Lemma 6.14. By Proposition 3.1 in [14] and Proposition 6.1, we check that \tilde{R} is an indecomposable representation of \mathcal{D} . Consequently, \mathcal{D} is of infinite representation type. This contradiction shows that H must act transitively on at least one biset, proving the first statement.

Now we turn to the proof of the second statement. There are three cases:

Case I: \mathcal{D} is the first category. We consider the full subcategory \mathcal{E} with objects x and z . Note that $\mathcal{E}(x, z) = L\beta H\alpha G$. Since \mathcal{E} is of finite representation type as well, either L or G should act transitively on $\mathcal{E}(x, z)$. Without loss of generality suppose that L acts transitively, i.e., $L\beta H\alpha G = L\beta\alpha$.

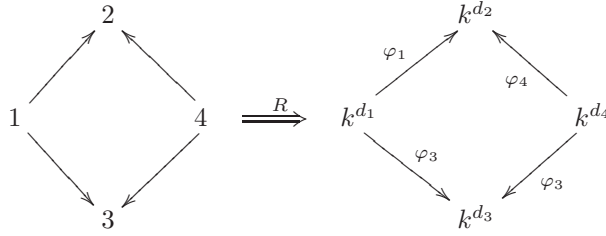
Take an arbitrary $h \in H$ and consider $\beta h\alpha$. Since $\beta h\alpha \in \mathcal{E}(x, y) = L\beta\alpha$, we can find some $l \in L$ such that $\beta h\alpha = l\beta\alpha$. Consequently, this non-isomorphism in $\mathcal{C}(x, z)$ has two decompositions into unfactorizable morphisms. By the unique factorization property, there should exist $h_1 \in H$ such that the following diagram commutes:

$$\begin{array}{ccccc}
 x & \xrightarrow{\alpha} & y & \xrightarrow{l\beta} & z \\
 \downarrow 1 & & \downarrow h_1 & & \downarrow 1 \\
 x & \xrightarrow{h\alpha} & y & \xrightarrow{\beta} & z
 \end{array}$$

Consequently, $l\beta = \beta h_1$ and $h_1\alpha = h\alpha$. So $h_1 \in H_2$ and $h^{-1}h_1 \in H_1$. Therefore, $h \in H_2H_1$, and $|H_1 \setminus H/H_2| = 1$.

In the next two cases we consider the second category. Suppose that $|H_1 \setminus H/H_2| \geq 2$ (so $H_1 \neq H \neq H_2$), we want to get a contradiction.

Case II: $k \uparrow_{H_1}^H$ and $k \uparrow_{H_2}^H$ has a common composition factor $S \not\cong k$. Then the projective kH -module $P_S \not\cong P_k$ satisfies that $\text{Top}(P_S \downarrow_{H_1}^H)$ and $\text{Soc}(P_S \downarrow_{H_2}^H)$ have a summand isomorphic to k_{H_1} and k_{H_2} respectively. For an indecomposable representation R of the following quiver with dimension vector (d_1, d_2, d_3, d_4) ,



we can construct a corresponding representation \tilde{R} of \mathcal{D} as follows: $\tilde{R}(x) \cong k^{d_1}$, $\tilde{R}(y) \cong k^{d_2} \oplus P_S^{d_3}$, $\tilde{R}(z) \cong k^{d_4}$. Linear maps $\tilde{R}(\alpha)$ and $\tilde{R}(\beta)$ can be defined in a way as we did in the proof of Lemma 6.14. We deduce that \mathcal{D} is of infinite representation type, contradicting the given condition.

Case III: $k \uparrow_{H_1}^H$ and $k \uparrow_{H_2}^H$ has no common composition factors except k . By the previous lemma, the projective kH -module P_k and the Scott modules of both $k \uparrow_{H_1}^H$ and $k \uparrow_{H_2}^H$ only have composition factors isomorphic to k . Moreover, their dimensions are all bigger than 1.

Consider the following category \mathcal{E} :

$$\mathcal{E} : \quad \begin{array}{c} \textcircled{\curvearrowright} x \xrightarrow{H\bar{\alpha}} y \xleftarrow{H\bar{\beta}} z \textcircled{\curvearrowright} \\ \quad \quad \quad \uparrow \textcircled{\curvearrowright} \\ \quad \quad \quad H \end{array}$$

where $\text{Stab}_H(\bar{\alpha}) = H_1$ and $\text{Stab}_H(\bar{\beta}) = H_2$. The reader can check that \mathcal{E} is a quotient category of \mathcal{D} . So it suffices to prove the infinite representation type of \mathcal{E} .

Let e be a primitive idempotent in kH such that $kHe \cong P_k$. Consider the algebra

$$\Lambda = \text{End}_{k\mathcal{E}}(k\mathcal{E}1_x \oplus k\mathcal{E}e \oplus k\mathcal{E}1_z)^{\text{op}} \cong (1_x + e + 1_z)k\mathcal{E}(1_x + e + 1_z).$$

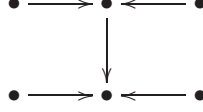
It is easy to check that $1_x k\mathcal{E}1_x \cong k \cong 1_z k\mathcal{E}1_z$, $ek\mathcal{E}e \cong k[t]/(t^d)$ where $d = \dim_k P_k \geq 2$, and

$$\dim_k ek\mathcal{E}1_x = \dim_k ekH\alpha = \dim_k \text{Hom}_{kH}(P_k, k \uparrow_{H_1}^H) = s \geq 2,$$

where s is the dimension of the Scott module of $k \uparrow_{H_1}^H$. Similarly, $\dim_k ek\mathcal{E}1_z = t \geq 2$, where t is the dimension of the Scott module of $k \uparrow_{H_2}^H$. Therefore, Λ is isomorphic to the path algebra of the following bounded quiver



with relations $\delta^d = \delta^s \gamma = \delta^t \mu = 0$. Its covering is not locally representation-finite since it contains the following subquiver:



Therefore, Λ and hence \mathcal{E} are of infinite representation type, contradicting the given condition. \square

Applying the same technique, we can show the following lemma:

Lemma 8.4. *Let \mathcal{D} be a finite free EI category for which the underlying quiver Q is a Dynkin quiver of type D_4 . Let x be the object with three adjacent objects. If \mathcal{D} is of finite representation type and $\text{char}(k) = p \neq 2, 3$, then $\mathcal{D}(x, x)$ acts transitively on all bisets of unfactorizable morphisms in this category.*

Proof. If $\mathcal{D}(x, x)$ does not act transitively on some biset, then we can construct infinitely many non-isomorphic indecomposable representations for \mathcal{D} using the first quiver shown in the proof of the previous lemma. We omit the details. \square

We are ready to prove conditions (5) and (6) in Theorem 1.4.

Proposition 8.5. *Let \mathcal{C} be a connected, skeletal finite free EI category with finite representation type. Suppose that $\text{char}(k) = p \neq 2, 3$. Then the following conditions must hold:*

- (1) *If there are two representative unfactorizable morphisms starting or ending at $x \in \text{Ob } \mathcal{C}$, then $\mathcal{C}(x, x)$ acts transitively on at least one biset generated by them.*
- (2) *Let y be the unique object (if exists) at which three representative unfactorizable morphisms start or end. Then $\mathcal{C}(y, y)$ acts transitively on all bisets generated by them.*

Since the underlying quiver of \mathcal{C} is a Dynkin quiver, every object $x \in \text{Ob } \mathcal{C}$ can have at most three representative unfactorizable morphisms ending or starting at it.

Proof. We prove the first statement. Denote these two representative unfactorizable morphism by α and β , and let \mathcal{D} be the full subcategory constituted of sources and targets of α and β . Then either \mathcal{D} or \mathcal{D}^{op} is one of the two categories shown before Proposition 8.3. Applying this lemma to \mathcal{D} or \mathcal{D}^{op} we get the conclusion. The second statement can be proved using Lemma 8.4. \square

REFERENCES

- [1] J. Alperin, *Local representation theory*, Cambridge Studies in Adv. Math. 11, Cambridge University Press, (1986).
- [2] M. Auslander, I. Reiten, and S. Smalø, *Representation theory of Artin algebras*, Cambridge Studies in Advanced Mathematics (36), Cambridge University Press, (1997);
- [3] R. Bautista, P. Gabriel, A.V. Roiter and L. Salmerón, *Representation-finite algebras and multiplicative bases*, Invent. math. 81, 217-285, (1985);
- [4] D. J. Benson, *Representations and cohomology I: Basic representation theory of finite groups and associative algebras*, 2nd edition, Cambridge Studies in Advanced Mathematics 30, Cambridge University Press, (1998);

- [5] K. Bongartz and P. Gabriel, *Covering spaces in representation theory*, Invent. math. 65, 331-378, (1982); 217-285, (1985);
- [6] S. Bouc, *Foncteurs d'ensembles munis d'une double action*, J. of Algebra, 183, 664-736, (1996);
- [7] M. Broué, *On Scott modules and p -permutation modules: an approach through the Brauer morphism*, Proc. Amer. Math. Soc. 93, 401-408, (1985);
- [8] M. Butler, and C. Ringel, *Auslander-Reiten sequences with few middle terms and applications to string algebras*, Comm. Algebra 15, 145-179, (1987);
- [9] T. Dieck, *Transformation groups*, de Gruyter Studies in Math. 8, Walter de Gruyter, (1987);
- [10] K. Dietrich, *Representation theory of EI categories*, Ph.D. Thesis, University of Paderborn, (2010);
- [11] K. Dietrich, *Representation types of EI categories*, preprint, arXiv: 1104.2339;
- [12] E. Green, *Graphs with relations, coverings and group-graded algebras*, Tran. AMS, 29, 297-310, (1983);
- [13] J. A. Green, *Multiplicities, Scott modules and lower defect groups*, J. London Math. Soc. (2), 28, 282-292, (1983);
- [14] L. Li, *A chracterization of finite EI categories with hereditary category algebras*, J. Algebra 345, 213-241, (2011);
- [15] L. Li, *A generalized Koszul theory and its application*, accepted by Tran. AMS, arXiv: 1109.5760;
- [16] M. Loupias, *Indecomposable representations of finite ordered sets*, Springer, Lecture Notes in Mathematics 488, 201-209, (1975);
- [17] W. Lück, *Transformation groups and algebraic K-theory*, Lecture Notes in Mathematics 1408, Springer-Verlag, (1989);
- [18] S. Mac Lane, *Categories for the Working Mathematician*, GTM 5, Springer-Verlag, (1971);
- [19] B. Mitchell, *Rings with several objects*, Advances in Math. 8, 1-161, (1972);
- [20] M. Takahashi, *Scott modules in finite groups with cyclic Sylow p -subgroups*, J. Algebra 353, 298-318, (2012);
- [21] P. Webb, *A guide to Mackey functors*, pages 805-836 in Handbook of Algebra vol 2 (ed M. Hazewinkel), Elsevier, (2000);
- [22] P. Webb, *An introduction to the representations and cohomology of categories*, Group Representation Theory, M. Geck, D. Testerman and J. Thvenaz(eds), EPFL Press (Lausanne), 149-173, (2007);
- [23] P. Webb, *Standard stratifications of EI categories and Alperin's weight conjecture*, J. Algebra 320, 4073-4091, (2008);
- [24] F. Xu, *Homological properties of category algebras*, Ph.D. Thesis, University of Minnesota, (2006);
- [25] F. Xu, *Representations of categories and their applications*, J. of Algebra, 317, 153-183, (2007).

SCHOOL OF MATHEMATICS, UNIVERSITY OF MINNESOTA, MINNEAPOLIS, MN, 55455, USA
E-mail address: lixx480@math.umn.edu